Math 154: Midterm

1. Properties of the Gaussian distribution (10pt)

(1) Let $X \sim N(0, \sigma^2)$, i.e. it is a continuous random variable with probability density function
$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}.$$
Show that $\mathbb{E}[X^k] = 0$ for any odd $k \geqslant 1$.

(2) Show that for any $\lambda \in \mathbb{R}$, we have $\mathbb{E}e^{\lambda X} = e^{\frac{\lambda^2\sigma^2}{2}}$.

(3) Compute $\mathbb{E}[X^2]$ using this moment generating function formula.

(4) Now, suppose $\sigma = 1$. You can take for granted that for any smooth function $f : \mathbb{R} \to \mathbb{R}$, we have $\mathbb{E}[Xf(X)] = \mathbb{E}[f'(X)]$. Compute $\mathbb{E}[X^4]$ using this formula.

(5) Show that if $X \sim N(0, \sigma_1^2)$ and $Y \sim N(0, \sigma_2^2)$ for $\sigma_1, \sigma_2 > 0$, and that if X, Y are independent, then $X + Y \sim N(0, \sigma_1^2 + \sigma_2^2)$.

Solution:

(1) We have $\mathbb{E}[X^k] = \int_{\mathbb{R}} x^k p(x)\mathrm{d}x$. But $x^k p(x)$ is odd since $p(x)$ is even (this is for k odd). Thus, $\mathbb{E}[X^k] = 0$.

(2) We have
$$\mathbb{E}e^{\lambda X} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\sigma^2} + \lambda x}\mathrm{d}x = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-\frac{(x-\lambda\sigma^2)^2}{2\sigma^2}} e^{\frac{\lambda^2\sigma^2}{2}}\mathrm{d}x = e^{\frac{\lambda^2 x^2}{2}},$$
where the last step follows because $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\lambda\sigma^2)^2}{2\sigma^2}}$ is the pdf of $N(\lambda\sigma^2, \sigma^2)$, so its integral is 1.

(3) We have
$$\mathbb{E}[X^2] = \partial_\lambda^2 e^{\frac{\lambda^2\sigma^2}{2}}|_{\lambda=0} = \partial_\lambda \left(\lambda\sigma^2 e^{\frac{\lambda^2\sigma^2}{2}}\right)|_{\lambda=0} = \sigma^2 e^{\frac{\lambda^2\sigma^2}{2}}|_{\lambda=0} + \lambda^2\sigma^4 e^{\frac{\lambda^2\sigma^2}{2}}|_{\lambda=0} = \sigma^2.$$

(4) Choose $f(X) = X^3$, so that $\mathbb{E}[X^4] = \mathbb{E}[Xf(X)] = \mathbb{E}[f'(X)] = 3\mathbb{E}[X^2] = 3\sigma^2$.

(5) We have $\mathbb{E}e^{\lambda(X+Y)} = \mathbb{E}e^{\lambda X}\mathbb{E}e^{\lambda Y} = e^{\frac{\lambda^2\sigma_1^2}{2}} e^{\frac{\lambda^2\sigma_2^2}{2}} = e^{\frac{\lambda^2(\sigma_1^2+\sigma_2^2)}{2}}$ by independence and part (2). This is the MGF for $N(0, \sigma_1^2 + \sigma_2^2)$, so by inversion theorem, we are done.

2. PROPERTIES OF THE POISSON DISTRIBUTION (10PT)

(1) Suppose $X \sim \mathrm{Pois}(\lambda)$, i.e. it is a discrete random variable with mass function $p(k) = \frac{\lambda^k}{k!}e^{-\lambda}$ if $k \geqslant 0$ is an integer and $p(x) = 0$ else. Show that $\sum_{k=0}^{\infty} p(k) = 1$, so that it is indeed a probability mass function.
(2) Show $\mathbb{E}e^{\xi X} = e^{\lambda(e^{\xi}-1)}$ for any $\xi \in \mathbb{R}$.
(3) Compute $\mathbb{E}[X^k]$ for $k = 1, 2, 3, 4$.
(4) Suppose $Y \sim \mathrm{Pois}(\mu)$ and X, Y are independent. Show that $X + Y \sim \mathrm{Pois}(\lambda + \mu)$ using the moment generating function.
(5) Give another proof of point (4) using the convolution formula. (If you do not remember the convolution formula, try to use the law of total probability to compute the mass function of $X + Y$ by conditioning on the value of X.)

Solution:

(1) We have $\sum_{k=0}^{\infty} p(k) = e^{-\lambda}\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda} = 1$.
(2) We have $\mathbb{E}e^{\xi X} = \sum_{k=0}^{\infty} \frac{\lambda^k e^{k\xi}}{k!}e^{-\lambda} = e^{-\lambda}\sum_{k=0}^{\infty} \frac{(\lambda e^{\xi})^k}{k!} = e^{-\lambda}e^{\lambda e^{\xi}} = e^{\lambda(e^{\xi}-1)}$.
(3) We have

$$\begin{aligned}
\partial_\xi e^{\lambda(e^\xi-1)} &= \lambda e^\xi e^{\lambda(e^\xi-1)}, \\
\partial_\xi^2 e^{\lambda(e^\xi-1)} &= \lambda e^\xi e^{\lambda(e^\xi-1)} + \lambda^2 e^{2\xi} e^{\lambda(e^\xi-1)}, \\
\partial_\xi^3 e^{\lambda(e^\xi-1)} &= \lambda e^\xi e^{\lambda(e^\xi-1)} + \lambda^2 e^{2\xi} e^{\lambda(e^\xi-1)} + 2\lambda^2 e^{2\xi} e^{\lambda(e^\xi-1)} + \lambda^3 e^{3\xi} e^{\lambda(e^\xi-1)} \\
&= \lambda e^\xi e^{\lambda(e^\xi-1)} + 3\lambda^2 e^{2\xi} e^{\lambda(e^\xi-1)} + \lambda^3 e^{3\xi} e^{\lambda(e^\xi-1)}, \\
\partial_\xi^4 e^{\lambda(e^\xi-1)} &= \lambda e^\xi e^{\lambda(e^\xi-1)} + \lambda^2 e^{2\xi} e^{\lambda(e^\xi-1)} + 6\lambda^2 e^{2\xi} e^{\lambda(e^\xi-1)} + 3\lambda^3 e^{3\xi} e^{\lambda(e^\xi-1)} \\
&\quad + 3\lambda^3 e^{3\xi} e^{\lambda(e^\xi-1)} + \lambda^4 e^{4\xi} e^{\lambda(e^\xi-1)}.
\end{aligned}$$

Now, set $\xi = 0$ to get

$$\begin{aligned}
\mathbb{E}X &= \lambda, \\
\mathbb{E}X^2 &= \lambda + \lambda^2, \\
\mathbb{E}X^3 &= \lambda + 3\lambda^2 + \lambda^3, \\
\mathbb{E}X^4 &= \lambda + 7\lambda^2 + 6\lambda^3 + \lambda^4.
\end{aligned}$$

(4) We have $\mathbb{E}e^{\xi(X+Y)} = \mathbb{E}e^{\xi X}\mathbb{E}e^{\xi Y} = e^{\lambda(e^\xi-1)}e^{\mu(e^\xi-1)} = e^{(\lambda+\mu)(e^\xi-1)}$ by independence and point (2). By the inversion theorem, and because this is the MGF of $\mathrm{Pois}(\lambda+\mu)$ by point (2), we are done.

(5) By the convolution formula, we have

$$
\begin{aligned}
\mathbb{P}(X+Y=k) &= \sum_{\ell} \mathbb{P}(X=k-\ell)\mathbb{P}(Y=\ell) \\
&= \sum_{\ell=0}^{k} e^{-\lambda} e^{-\mu} \frac{\lambda^{k-\ell}}{(k-\ell)!} \frac{\mu^{\ell}}{\ell!} \\
&= \frac{e^{-\lambda-\mu}}{k!} \sum_{\ell=0}^{k} \frac{k!}{(k-\ell)!\ell!} \lambda^{k-\ell} \mu^{\ell} \\
&= \frac{(\lambda+\mu)^k}{k!} e^{-\lambda-\mu},
\end{aligned}
$$

which is the pmf for $\mathrm{Pois}(\lambda+\mu)$.

3. Some properties of the exponential distribution (10pt)

(1) Suppose $X \sim \mathrm{Exp}(\lambda)$, i.e. it is a continuous random variable with probability density function $p(x) = \lambda e^{-\lambda x} \mathbf{1}_{x \geqslant 0}$. Compute $\mathbb{E}[X^k]$ for $k = 1, 2, 3$.

(2) Show that $\mathbb{P}[X \geqslant x + s | X \geqslant s] = \mathbb{P}[X \geqslant x]$ for any $x, s \geqslant 0$.

(3) Is the sum of two independent exponential random variables also an exponential random variable? Why or why not?

(4) Fix any $t > 0$, and suppose $Y \sim \mathrm{Pois}(t\lambda)$. Show that $\mathbb{P}[Y = 0] = \mathbb{P}[X \geqslant t]$. This is the Poisson-exponential duality.

Solution:

(1) We first compute $\mathbb{E}e^{\xi X} = \int_0^\infty \lambda e^{-(\lambda - \xi)x} \mathrm{d}x = \frac{\lambda}{\lambda - \xi}$. Next, we compute

$$\begin{aligned}
\partial_\xi \frac{\lambda}{\lambda - \xi} &= \frac{\lambda}{(\lambda - \xi)^2}, \\
\partial_\xi^2 \frac{\lambda}{\lambda - \xi} &= 2\frac{\lambda}{(\lambda - \xi)^3}, \\
\partial_\xi^3 \frac{\lambda}{\lambda - \xi} &= 6\frac{\lambda}{(\lambda - \xi)^4}.
\end{aligned}$$

If we set $\xi = 0$, we get $\mathbb{E}X = \lambda^{-1}$ and $\mathbb{E}X^2 = 2\lambda^{-2}$ and $\mathbb{E}X^3 = 6\lambda^{-3}$.

(2) By definition of conditional probability, we have

$$\begin{aligned}
\mathbb{P}[X \geqslant x + s | X \geqslant s] &= \frac{\mathbb{P}[X \geqslant x + s, X \geqslant s]}{\mathbb{P}[X \geqslant s]} = \frac{\mathbb{P}[X \geqslant x + s]}{\mathbb{P}[X \geqslant s]} \\
&= \frac{\int_{x+s}^\infty \lambda e^{-\lambda u} \mathrm{d}u}{\int_s^\infty \lambda e^{-\lambda u} \mathrm{d}u} = \frac{e^{-\lambda(x+s)}}{e^{-\lambda s}} \\
&= e^{-\lambda x} = \int_x^\infty \lambda e^{-\lambda u} \mathrm{d}u = \mathbb{P}[X \geqslant x].
\end{aligned}$$

(3) No. The moment generating function of the sum of independent $\mathrm{Exp}(\lambda)$ is $\frac{\lambda^2}{(\lambda - \xi)^2}$, which is not of the form $\frac{\mu}{\mu - \xi}$ for any $\mu \in \mathbb{R}$.

(4) We already showed in part (2) that $\mathbb{P}[X \geqslant t] = e^{-t\lambda}$. But $\mathbb{P}[Y = 0] = e^{-t\lambda}$ as well by the pmf in Problem 2.

4. A PROBLEM ABOUT COINS (OF COURSE) (10PT)

(1) Take a coin that lands heads with probability p and tails with probability $1-p$. Let q_n be the probability that an even number of heads have been tossed after the n-th flip. (The flips are jointly independent, and 0 is even.) Show that $q_0 = 1$ and

$$q_n = p(1 - q_{n-1}) + (1 - p)q_{n-1}.$$

(2) Find (with proof that it is correct) a formula for q_n in terms of n and p only.

Solution:

(1) Clearly $q_0 = 1$, since no tosses means zero heads. Now, for $n \geqslant 1$, an even number of heads after n flips means one of the following. The n-th toss was a head, and the number of heads before the n-th flip was odd. Or, the n-th toss was a tail, and the number of heads before the n-th flip was even. In particular, if $\mathcal{E}_{even,k}$ is the event where an even number of heads are flipped by time k and $\mathcal{E}_{odd,k}$ is the complement event, we have

$$\begin{aligned} q_n &= \mathbb{P}(X_n = H, \mathcal{E}_{odd,N-1}) + \mathbb{P}(X_n = T, \mathcal{E}_{even,N-1}) \\ &= \mathbb{P}(X_n = H)\mathbb{P}(\mathcal{E}_{odd,N-1}) + \mathbb{P}(X_n = T)\mathbb{P}(\mathcal{E}_{even,N-1}), \end{aligned}$$

where the second identity follows because the flips are jointly independent. Note that $\mathbb{P}(X_n = H) = p$ and $\mathbb{P}(X_n = T) = 1 - p$. Moreover, $\mathbb{P}(\mathcal{E}_{even,k}) = q_k$ and $\mathbb{P}(\mathcal{E}_{odd,k}) = 1 - q_k$. This finishes the proof.

(2) By rearranging, we have $q_n = p + (1 - 2p)q_{n-1}$. We claim that the solution to this recurrence relation, given $q_0 = 1$, is $q_n = \frac{1}{2} + \frac{1}{2}(1 - 2p)^n$. Indeed, we have $q_0 = \frac{1}{2} + \frac{1}{2}(1 - 2p)^0 = \frac{1}{2} + \frac{1}{2} = 1$. Moreover, we can check

$$p + (1 - 2p)\left(\frac{1}{2} + \frac{1}{2}[1 - 2p]^{n-1}\right) = p + \frac{1}{2}(1 - 2p) + \frac{1}{2}[1 - 2p]^n = \frac{1}{2} + \frac{1}{2}[1 - 2p]^n,$$

so that this choice of q_n solves the desired recurrence relation with the appropriate condition $q_0 = 1$.

5. SUGGESTED READING (2PT)

(1) Say a little about Joseph Doob.

Solution: I said quite a bit in class, so I encourage you to read the Wikipedia page.