Math 154: Probability Theory, HW 9

DUE APRIL 16, 2024 BY 9AM

Remember, if you are stuck, take a look at the lemmas/theorems/examples from class, and see if anything looks familiar.

1. GETTING OUR HANDS ON BROWNIAN MOTION

1.1. **A computation.** Consider the integral $\int_0^t \mathbf{B}_s^2 \mathrm{d}s$.

(1) Compute $\mathbb{E}\int_0^t \mathbf{B}_s^2 \mathrm{d}s$.
(2) Compute $\mathbb{E}|\int_0^t \mathbf{B}_s^2 \mathrm{d}s|^2$. (*Hint*: as in class, square the integral to get a double integral over $0 \leqslant r \leqslant s \leqslant t$. For $r \leqslant s$, it may then help to write $\mathbf{B}_s^2\mathbf{B}_r^2 = (\mathbf{B}_s - \mathbf{B}_r + \mathbf{B}_r)^2\mathbf{B}_r^2 = (\mathbf{B}_s - \mathbf{B}_r)^2\mathbf{B}_r^2 + 2(\mathbf{B}_s - \mathbf{B}_r)\mathbf{B}_r^3 + \mathbf{B}_r^4$. Now use independence of increments and knowledge of the distribution of increments.)
(3) Deduce the variance of $\int_0^t \mathbf{B}_s^2 \mathrm{d}s$.

Solution. (1) We have $\mathbb{E}\int_0^t \mathbf{B}_s^2 \mathrm{d}s = \int_0^t \mathbb{E}\mathbf{B}_s^2 \mathrm{d}s = \int_0^t s\mathrm{d}s = \frac{1}{2}t^2$.

(2) We have

$$\begin{aligned}\mathbb{E}\left|\int_0^t \mathbf{B}_s^2 \mathrm{d}s\right|^2 &= \int_0^t\int_0^t \mathbb{E}[\mathbf{B}_s^2\mathbf{B}_r^2]\mathrm{d}r\mathrm{d}s = 2\int_0^t\int_0^s \mathbb{E}[\mathbf{B}_s^2\mathbf{B}_r^2]\mathrm{d}r\mathrm{d}s \\ &= 2\int_0^t\int_0^s \mathbb{E}[(\mathbf{B}_s - \mathbf{B}_r)^2\mathbf{B}_r^2]\mathrm{d}r\mathrm{d}s + 4\int_0^t\int_0^s \mathbb{E}[(\mathbf{B}_s - \mathbf{B}_r)\mathbf{B}_r^3]\mathrm{d}r\mathrm{d}s \\ &+ 2\int_0^t\int_0^s \mathbb{E}[\mathbf{B}_r^4]\mathrm{d}r\mathrm{d}s.\end{aligned}$$

The second term in the last expression is zero by independence and mean-zero of increments. Since $\mathbf{B}_r \sim N(0,r)$ and $\mathbf{B}_s - \mathbf{B}_r \sim N(0, s-r)$, by independent of increments, we have

$$\begin{aligned}2\int_0^t\int_0^s \mathbb{E}[(\mathbf{B}_s - \mathbf{B}_r)^2\mathbf{B}_r^2]\mathrm{d}r\mathrm{d}s &= 2\int_0^t\int_0^s (s-r)r\mathrm{d}r\mathrm{d}s \\ &= 2\int_0^t\int_0^s sr\mathrm{d}r\mathrm{d}s - 2\int_0^t\int_0^s r^2\mathrm{d}r\mathrm{d}s \\ &= \int_0^t s^3\mathrm{d}s - \frac{2}{3}\int_0^t s^3\mathrm{d}s = \frac{1}{4}t^4 - \frac{1}{6}t^4 = \frac{1}{12}t^4, \\ 2\int_0^t\int_0^s \mathbb{E}[\mathbf{B}_r^4]\mathrm{d}r = 6\int_0^t\int_0^s r^2\mathrm{d}r\mathrm{d}s &= 2\int_0^t s^3\mathrm{d}s = \frac{1}{2}t^4.\end{aligned}$$

By combining the previous two displays, we get $\mathbb{E}|\int_0^t \mathbf{B}_s^2 \mathrm{d}s|^2 = \frac{7}{12}t^4$.

(3) By parts (1) and (2), we have $\mathrm{Var}\int_0^t \mathbf{B}_s^2 \mathrm{d}s = \frac{7}{12}t^4 - \frac{1}{4}t^4 = \frac{1}{3}t^4$.

□

1.2. **Brownian Gambler's ruin (*Hint*: use optional stopping!)** Let $\mathbf{B}$ be Brownian motion, and fix $a, b > 0$. Let $\tau_{a,b}$ be the first time τ such that $\mathbf{B}_\tau \in \{-a, b\}$.

(1) Find the probability that $\mathbf{B}_{\tau_{a,b}} = a$.
(2) Compute $\mathbb{E}\tau_{a,b}$.

Solution. (1) By optional stopping and the martingale property of $\mathbf{B}$, we have $\mathbb{E}\mathbf{B}_{\tau_{a,b}} = 0$. But $\mathbb{E}\mathbf{B}_{\tau_{a,b}} = -a\mathbb{P}[\mathbf{B}_{\tau_{a,b}} = -a] + b\mathbb{P}[\mathbf{B}_{\tau_{a,b}} = b]) = -a\mathbb{P}[\mathbf{B}_{\tau_{a,b}} = -a] + b(1 - \mathbb{P}[\mathbf{B}_{\tau_{a,b}} = -a])$. Thus, we get $\mathbb{P}[\mathbf{B}_{\tau_{a,b}} = -a] = \frac{b}{a+b}$.

(2) By optional stopping and the martingale property of $\mathbf{B}_t^2 - t$, we have $\mathbb{E}\mathbf{B}_{\tau_{a,b}}^2 = \mathbb{E}\tau_{a,b}$. By part (1), we have $\mathbb{E}\tau_{a,b} = \mathbb{E}\mathbf{B}_{\tau_{a,b}}^2 = a^2\frac{b}{a+b} + b^2\frac{a}{a+b} = \frac{a^2b+ab^2}{a+b}$.

□

1.3. **Moment generating function of Gaussians, Brownian motion style.** Consider the process $\mathbf{M}_t := \exp\{\lambda \mathbf{B}_t - \mu t\}$, where $\lambda, \mu \in \mathbb{R}$.

(1) Fix $\lambda \in \mathbb{R}$. For which $\mu = \mu(\lambda) \in \mathbb{R}$ does $\mathbf{M}$ satisfy the martingale property? ($\mu(\lambda)$ will depend on λ.) In what follows, we will always take $\mathbf{M}_t$ for this choice of $\mu = \mu(\lambda)$.

(2) Fix $\lambda \in \mathbb{R}$. Show that $\mathbb{E}\mathbf{M}_1 = 1$.

(3) Deduce that if $Z \sim N(0,1)$, then $\mathbb{E}e^{\lambda Z} = e^{\lambda^2/2}$. (*Hint*: recall $\mathbf{B}_1 \sim N(0,1)$.)

Solution. (1) As shown in class, for $\mathbf{M}_t$ to be a martingale, we need to find μ such that

$$\left(\partial_t + \frac{1}{2}\partial_x^2\right)\exp\{\lambda x - \mu t\} = 0.$$

The LHS is equal to $\exp\{\lambda x - \mu t\}(-\mu + \frac{1}{2}\lambda^2)$. Thus, it suffices to take $\mu = \frac{1}{2}\lambda^2$.

(2) By the martingale property, we have $\mathbb{E}\mathbf{M}_1 = \mathbb{E}\mathbf{M}_0 = 1$.

(3) By part (2), we have $\mathbb{E}e^{\lambda \mathbf{B}_1 - \lambda^2/2} = 1$. Thus, $\mathbb{E}e^{\lambda \mathbf{B}_1} = e^{\lambda^2/2}$. Conclude by the hint.

□

1.4. **Ergodicity of the OU process.** Suppose X_t is an OU process with initial condition X_0, that is $\mathrm{d}X_t = -X_t\mathrm{d}t + \mathrm{d}\mathbf{B}_t$, where $\mathbf{B}_t$ is a Brownian motion.

(1) Show that $N(0,1)$ is an invariant distribution for the OU process (see the notes for what this means).

(2) Let Z_t be an OU process with initial condition $Z_0 \sim N(0,1)$. That is, $\mathrm{d}Z_t = -Z_t + \mathrm{d}\mathbf{B}_t$, where $\mathbf{B}$ is the *same* Brownian motion from above. Define $Y_t = X_t - Z_t$. Show that $Y_t = Y_0 e^{-t}$ for all $t \geqslant 0$. Deduce that $Y_t \to 0$ as $t \to \infty$. (*Hint*: compute the differential equation solved by Y_t using the SDEs for X_t, Z_t; you can use that any solution to $f'(t) = -f(t)$ is given by $f(t) = f(0)e^{-t}$.)

Solution. (1) As shown in class, it suffices to show that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}p(x) + \frac{\mathrm{d}}{\mathrm{d}x}\left(xp(x)\right) = 0,$$

where $p(x)$ is the pdf for $N(0,1)$. We check this directly:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}p(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} = -\frac{1}{\sqrt{2\pi}}xe^{-\frac{x^2}{2}},\\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}p(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{\sqrt{2\pi}}xe^{-\frac{x^2}{2}}\right) = -\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} + \frac{1}{\sqrt{2\pi}}x^2e^{-\frac{x^2}{2}}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2}p(x) + \frac{\mathrm{d}}{\mathrm{d}x}\left(xp(x)\right) &= -\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} + \frac{1}{\sqrt{2\pi}}x^2e^{-\frac{x^2}{2}} - \frac{1}{\sqrt{2\pi}}x^2e^{-\frac{x^2}{2}} + \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\\ &= 0,\end{aligned}$$

so we are done.

(2) We have $\mathrm{d}Y_t = \mathrm{d}X_t - \mathrm{d}Z_t = -X_t\mathrm{d}t + Z_t\mathrm{d}t = -Y_t\mathrm{d}t$. Now use the hint to get $Y_t = Y_0 e^{-t}$.

□

1.5. **Brownian bridge.** The Brownian bridge is a "Brownian motion conditioned to hit 0 at time 1". The point of this exercise is to make this precise in a more natural way.

Let $\{z_k\}_{k=1}^{\infty}$ be a collection of i.i.d. $N(0,1)$ random variables. For any $N > 0$, define

$$\mathbf{Z}_t^{(N)} := \sum_{k=1}^{N} \frac{z_k\sqrt{2}}{k\pi}\sin(k\pi t).$$

Show that $\mathbf{Z}_0^{(N)} = \mathbf{Z}_1^{(N)} = 0$. Show that $\mathbb{E}\mathbf{Z}_t^{(N)} = 0$ and that

$$\mathbb{E}|\mathbf{Z}_t^{(N)} - \mathbf{Z}_t^{(M)}|^2 \to_{N,M\to\infty} 0.$$

Solution. We know that $\sin(k\pi) = 0$ for any integer k, so $\mathbf{Z}_0^{(N)}, \mathbf{Z}_1^{(N)} = 0$ follows. Since z_k have expectation 0, by linearity of expectation, we have $\mathbb{E}\mathbf{Z}_t^{(N)} = \sum_{k=1}^{N} \frac{\mathbb{E}[z_k]\sqrt{2}}{k\pi}\sin(k\pi t) = 0$. Moreover, we have

$$\mathbf{Z}_t^{(N)} - \mathbf{Z}_t^{(M)} = \sum_{k=N+1}^{M} \frac{z_k\sqrt{2}}{k\pi}\sin(k\pi t).$$

Since z_k are i.i.d. $N(0,1)$, we have

$$\mathbb{E}|\mathbf{Z}_t^{(N)} - \mathbf{Z}_t^{(M)}|^2 = \sum_{k=N+1}^{M} \frac{2\mathbb{E}|z_k|^2}{k^2\pi^2}\sin(k\pi t)^2 \leqslant \sum_{k=N+1}^{M} \frac{2}{k^2\pi^2},$$

which is $\leqslant CN^{-1}$ for some constant $C > 0$. Since $N^{-1} \to 0$ as $N \to \infty$, we are done. $\square$