Math 154: Probability Theory, HW 9

DUE APRIL 9, 2024 BY 9AM

Remember, if you are stuck, take a look at the lemmas/theorems/examples from class, and see if anything looks familiar.

1. SOME MORE PRACTICE WITH MARKOV CHAINS

1.1. **Concrete example.** Fix $\alpha, \beta \in (0,1)$, and let $\{X_n\}_n$ be the Markov chain with transition matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

(1) Compute the stationary distribution.
(2) For what values of $\alpha, \beta \in (0,1)$ is the Markov chain reversible in equilibrium?

1.2. **Sometimes it helps to be lazy.**

(1) Suppose $\{X_n\}_n$ is the Markov chain with state space $\{\mathrm{s}_1, \mathrm{s}_2, \mathrm{s}_3\}$ such that with probability 1, if $X_n = \mathrm{s}_k$, then $X_{n+1} = \mathrm{s}_{k+1}$ if $k < 3$ and $X_{n+1} = \mathrm{s}_1$ if $k = 3$ (in words, it keeps going right but loops back $3 \mapsto 1$). Write down its transition matrix.
(2) Let P be the transition matrix from part (1). For any $z \in \mathbb{C}$, compute $\det[z\mathrm{I}_3 - P]$, where I_3 is the 3×3 identity matrix and z is just multiplying every entry by z.
(3) In part (2), you should get a degree 3 polynomial. Show that its roots λ satisfy $|\lambda| = 1$.
(4) Now, let $\{Y_n\}_n$ be the Markov chain with state space $\{\mathrm{s}_1, \mathrm{s}_2, \mathrm{s}_3\}$ such that

$$\mathbb{P}[Y_{n+1} = \mathbf{s} | Y_n = \mathbf{s}_k] = \begin{cases} \frac{1}{2} & \mathbf{s} = \mathbf{s}_{k+1} \\ \frac{1}{2} & \mathbf{s} = \mathbf{s}_k \\ 0 & \text{else} \end{cases}$$

where $k+1$ is identified with 1 if $k = 3$. In words, Y goes to the right with probability $1/2$, and with probability $1/2$, it stays put. Show (directly without using any theorem about irreducibility from class) that its transition matrix has one eigenvalue $\lambda_1 = 1$, and its other eigenvalues satisfy $|\lambda| < 1$.

1.3. **Going backwards?** Let $\{X_n\}_n$ be the Markov chain with state space $\{A, B, C\}$ and transition matrix

$$P = \begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.6 & 0.2 & 0.2 \\ 0.4 & 0.2 & 0.4 \end{pmatrix}$$

Suppose that $X_0 = A$ with probability 1, just for simplicity.

(1) Compute the stationary distribution π for P (make sure the entries of π are non-negative and add up to 1). Show that P has eigenvalue 1 with multiplicity 1.

(2) Compute limits as $n \to \infty$ of $\mathbb{P}[X_n = s]$ for $s \in \{A, B, C\}$.
(3) Compute $\lim_{n\to\infty} \mathbb{P}[X_{n-1} = C | X_n = B]$. (*Hint*: use Bayes' rule)

1.4. **A linear algebra interpretation of reversibility.** Suppose P is a transition matrix of size $N \times N$ with stationary distribution π with entries $\pi_1, \dots, \pi_N$. For any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ with entries $\mathbf{x}_1, \dots, \mathbf{x}_N$ and $\mathbf{y}_1, \dots, \mathbf{y}_N$ respectively, define

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_k \mathbf{x}_k \mathbf{y}_k \pi_k.$$

Show P is reversible with respect to π if for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, we have $\langle \mathbf{x}, P\mathbf{y} \rangle = \langle P\mathbf{x}, \mathbf{y} \rangle$. (*Hint*: for showing reversibility of P, use the assumption $\langle \mathbf{x}, P\mathbf{y} \rangle = \langle P\mathbf{x}, \mathbf{y} \rangle$ with $\mathbf{x}, \mathbf{y}$ being standard basis vectors.)

1.5. **Brownian motion has the Markov property.** Recall that a Brownian motion $\{\mathbf{B}_t\}_{t \geqslant 0}$ satisfies the following.

- $\mathbf{B}_0$ with probability 1.
- For any times $0 < t_1 < t_2 < \ldots < t_{k+1}$, the vector $(\mathbf{B}_{t_1}, \mathbf{B}_{t_2} - \mathbf{B}_{t_1}, \dots, \mathbf{B}_{t_{k+1}} - \mathbf{B}_{t_k})$ has independent Gaussian entries with variance $\mathbb{E}|\mathbf{B}_{t_{\ell+1}} - \mathbf{B}_{t_\ell}|^2 = t_{\ell+1} - t_\ell$.

Show that for any sequence of times $0 < t_1 < t_2 < \ldots < t_{k+1}$ and any open set $A \subseteq \mathbb{R}$, we have

$$\mathbb{P}[\mathbf{B}_{t_{k+1}} \in A | \mathbf{B}_{t_k} = x_k, \dots, \mathbf{B}_{t_1} = x_1] = \mathbb{P}[\mathbf{B}_{t_{k+1}} \in A | \mathbf{B}_{t_k} = x_k].$$

(*Hint*: it may help to first explain why $\mathbf{B}_{t_{k+1}} - \mathbf{B}_{t_k}$ is independent of $\mathbf{B}_{t_\ell}$ for all $1 \leqslant \ell \leqslant k$.)