Math 154: Probability Theory, HW 8

DUE APRIL 2, 2024 BY 9AM

Remember, if you are stuck, take a look at the lemmas/theorems/examples from class, and see if anything looks familiar.

1. SOME PRACTICE WITH MARKOV CHAINS

1.1. **Classification of states.** Consider the state space $\{A, B, C, D\}$. For each Markov chain below (specified by its transition matrix), specify which states (i.e. which of A, B, C, D) are recurrent and which are transient. (Recall a transition matrix P has entries given by $P_{ij} = \mathbb{P}[i \to j]$.)

(1) $P_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(2) $P_2 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

(3) Find two row vectors π_1 and π_2 of length 4 such that $\pi_1 P_1 = \pi_1$ and $\pi_2 P_1 = \pi_2$. Your two row vectors cannot be scalar multiples of each (e.g. they must be linearly independent). What do you notice about the sign of each entry in π_1, π_2?

Solution. (1) States $1, 2, 4$ are recurrent. Indeed, $\{1, 2\}$ and $\{4\}$ are closed and communicating (since $\mathbb{P}[1 \to 2]$ and $\mathbb{P}[2 \to 1]$ are both positive). State 3 is transient, because the probability of entering a closed communicating class $\{4\}$ is positive.

(2) All states are recurrent. Indeed, $\mathbb{P}[A \to B]$ and $\mathbb{P}[B \to D]$ and $\mathbb{P}[D \to C]$ and $\mathbb{P}[C \to A]$ are all positive. Thus, there is a positive probability to go from A to any of B, C, D after a finite number of steps, and vice versa. This means the communicating class of A is all of $\{A, B, C, D\}$.

(3) Choose $\pi_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$ and $\pi_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}$. All entries are non-negative.

□

1.2. **A nice trick in computing long-time behavior of a Markov chain.** Consider P_1 from Problem 1.1. We will see that diagonalization from linear algebra is actually useful.

(1) Compute $\mathrm{Tr} P_1$ and $\det P_1$.

(2) Compute the eigenvalues of P_1. (*Hint*: the eigenvalues sum to the trace, and they multiply to the determinant. Use part (3) in Problem 1.1.)

(3) Label the eigenvalues as $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$, and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be the associated left eigenvectors, so that $\mathbf{v}_i P_1 = \lambda_i \mathbf{v}_i$. Note that $|\lambda_3|, |\lambda_4| < 1$. Deduce that for $i = 3, 4$, we have $\mathbf{v}_i P_1^n \to \vec{0}$ as $n \to \infty$, where $\vec{0} = (0, 0, 0, 0)$.

(4) Any vector $\mathbf{v}$ can be written as a linear combination $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 + \alpha_4 \mathbf{v}_4$. Show that $\mathbf{v} P_1^n \to \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2$ as $n \to \infty$. This shows that the long-time behavior of the P_1 Markov chain is rather simple!

Solution. (1) We have $\mathrm{Tr} P_1 = \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + 1 = \frac{9}{4}$ and $\det P_1 = 0$, since the first three columns of P_1 are linearly dependent.

(2) We know two eigenvalues $\lambda_1, \lambda_2 = 1$. Thus, the other eigenvalues satisfy $2 + \lambda_3 + \lambda_4 = \frac{9}{4}$ and one of λ_3, λ_4 is equal to 0. In particular, we have $\lambda_3 + 2 = \frac{9}{4}$, so $\lambda_3 = \frac{1}{4}$.

(3) We have $\mathbf{v}_3 P^n = \lambda_3 \mathbf{v}_3 P^{n-1} = \ldots = \lambda_3^n \mathbf{v}_3$. Since $|\lambda_3| < 1$, we have $\lambda_3^n \to 0$, so that $\mathbf{v}_3 P^n \to \vec{0}$. The same is true for λ_4, though in this case it is easier, since $\mathbf{v}_4 P^n = 0$ as long as $n \geqslant 1$.

(4) By linearity of matrix multiplication, we have $\mathbf{v} P_1^n = \alpha_1 \mathbf{v}_1 P_1^n + \alpha_2 \mathbf{v}_2 P_1^n + \alpha_3 \mathbf{v}_3 P_1^n + \alpha_4 \mathbf{v}_4 P_1^n$. Since $\mathbf{v}_j P_j = \mathbf{v}_j$ for $j = 1, 2$ (since $\lambda_1, \lambda_2 = 1$), we have $\mathbf{v} P_1^n = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 P_1^n + \alpha_4 \mathbf{v}_4 P_1^n$. Now use part (3).

□

1.3. **Random walk in dimension** 2**.** Let $\mathbf{X}(n) = (X_1(n), X_2(n))$, where X_1, X_2 are independent symmetric simple random walks such that $X_1(0), X_2(0) = 0$ and $n \geqslant 0$ is an integer. In what follows, you can use the computations from class.

(1) Show that for any $n \geqslant 0$, we have

$$\mathbb{P}[\mathbf{X}(2n) = (0,0)] = \binom{2n}{n}^2 2^{-4n}.$$

Deduce that $\mathbb{P}[\mathbf{X}(2n) = (0,0)] \geqslant Cn^{-1}$ for all $n \geqslant 1$, where $C \geqslant 0$ is some fixed constant. (*Hint*: use independence of X_1, X_2.)

(2) Show that

$$\sum_{n=1}^{\infty} \mathbb{P}[\mathbf{X}(n) = (0,0)] = \infty,$$

and deduce that $\mathbf{X}$ has the origin as a recurrent state.

Solution. (1) By independence of X_1, X_2, we have

$$\mathbb{P}[\mathbf{X}(2n) = (0,0)] = \mathbb{P}[X_1(2n) = 0, X_2(2n) = 0] = \prod_{i=1}^{2} \mathbb{P}[X_i(2n) = 0].$$

We computed in class that $\mathbb{P}[X_i(2n) = 0] = \binom{2n}{n} 2^{-2n}$, so the first claim follows. For the second, we computed in class that $\binom{2n}{n} 2^{-n} \geqslant C_1 n^{-1/2}$. Now square this.

(2) We have

$$\sum_{n=1}^{\infty} \mathbb{P}[\mathbf{X}(n) = 0, 0] \geqslant \sum_{n=1}^{\infty} \mathbb{P}[\mathbf{X}(2n) = (0,0)] \geqslant C \sum_{n=1}^{\infty} n^{-1},$$

and the last series diverges. Thus, the far LHS of the previous display must diverge as well, which means $\mathbf{X}$ has the origin as a recurrent state.

□

1.4. **Random walk in dimensions greater than or equal to 3.** Let $\mathbf{X}(n) = (X_1(n), \ldots, X_d(n))$, where $X_1, \ldots, X_d$ are independent symmetric simple random walks such that $X_1(0), \ldots, X_d(0) = 0$, and $n \geqslant 0$ is an integer and $d \geqslant 3$ is fixed.

(1) Show that $\mathbb{P}[\mathbf{X}(2n) = (0, \ldots, 0)] \leqslant Cn^{-d/2}$ for all $n \geqslant 1$, where C depends only on d.

(2) Show that $\mathbf{X}$ has the origin as a transient state if $d \geqslant 3$.

Solution. (1) Again, we have $\mathbb{P}[\mathbf{X}(2n) = (0, \ldots, 0)] = \mathbb{P}[\cap_{i=1}^d \{X_i(2n) = 0\}] = \prod_{i=1}^d \mathbb{P}[X_i(2n) = 0]$ by indepedence. We computed in class that $\mathbb{P}[X_i(2n) = 0] \leqslant C_1 n^{-1/2}$ for some $C_1 > 0$. Now take d-th powers.

(2) We have (by part (1)) that

$$\sum_{n=1}^{\infty} \mathbb{P}[\mathbf{X}(2n) = (0, \ldots, 0)] = \sum_{n=1}^{\infty} \mathbb{P}[\mathbf{X}(2n) = (0, \ldots, 0)] \leqslant C \sum_{n=1}^{\infty} n^{-d/2},$$

the last of which converges if $d/2 > 1$, hence if $d \geqslant 3$. The first identity above follows because a sum of k-many ± 1's cannot equal 0 if k is odd.

□

1.5. **Asymmetric simple random walk in dimension** 1**.** Suppose X is an asymmetric simple random walk on $\mathbb{Z}$. In particular,

$$\mathbb{P}[X(n+1) = x | X(n)] = \begin{cases} p & x = X(n)+1 \\ 1-p & x = X(n)-1 \\ 0 & \text{else} \end{cases}$$

where $p \neq \frac{1}{2}$. Suppose $X(0) = 0$. Define $S(n) = X(1) + \ldots + X(n)$ to be the random walk with $S(0) = 0$.

(1) Show that the process $M_n = S(n) - (2p-1)n$ is a martingale with respect to the sequence $\{X(k)\}_{k \geqslant 1}$. Show that $|M_{n+1} - M_n| \leqslant 4$ for all $n \geqslant 0$.
(2) Show that for some constant $C > 0$ independent of $n \geqslant 0$, we have
$$\mathbb{P}\left[|M_n| \geqslant n^{2/3}\right] \leqslant \exp\left\{-Cn^{1/3}\right\}$$
(3) Show that $\mathbb{P}[S(n) = 0] \leqslant \mathbb{P}[|M_n| \geqslant n^{2/3}]$ for n large enough. Using the bound $\exp\{-Cn^{1/3}\} \leqslant C_2 n^{-2}$ for some $C_2 > 0$ fixed, deduce that X has 0 as a transient state. (*Hint*: the assumption $p \neq \frac{1}{2}$ is crucial.)

Solution. (1) For any $n \geqslant 0$, we have

$$\begin{aligned}\mathbb{E}\left[M_{n+1} | X(1), \ldots, X(n)\right] &= \mathbb{E}\left[X(n+1) | X(1), \ldots, X(n)\right] + S(n) - (2p-1) - (2p-1)n \\ &= \mathbb{E}[X(n+1)] - (2p-1) + S(n) - (2p-1)n = M_n.\end{aligned}$$

The first identity follows by definition of M_{n+1} and linearity of conditional expectation (and that $S(n)$ is a function of $X(1), \ldots, X(n)$). The second identity follows by independence of $\{X(k)\}_k$. The last follows because $\mathbb{E}[X(k+1)] = p-(1-p) = 2p-1$. Finally, we note that $|M_{n+1}-M_n| = |2p-1-X(n+1)| \leqslant 2p+1+|X(n+1)| \leqslant 4$.

(2) By Azuma and part (1), we know that
$$\mathbb{P}[|M_n| \geqslant n^{2/3}] \leqslant \exp\left\{-C_1 \frac{n^{4/3}}{4n}\right\}$$
for some $C_1 > 0$. Now choose $C = C_1/4$.

(3) Note that $S(n) = 0$ implies $M_n = -(2p-1)n$. Thus, if n is large enough, say larger than some n_0, since $2p-1 \neq 0$ by assumtion, we know that $|(2p-1)n| \geqslant n^{2/3}$. This shows $\mathbb{P}[S(n) = 0] \leqslant \mathbb{P}[|M_n| \geqslant n^{2/3}]$. By the given bound and part (2), we have

$$\sum_{n=1}^{\infty} \mathbb{P}[S(n)=0] = \sum_{n=1}^{n_0} \mathbb{P}[S(n)=0] + \sum_{n=n_0+1}^{\infty} \mathbb{P}[S(n)=0] \leqslant n_0 + C_2 \sum_{n=n_0+1}^{\infty} n^{-2} < \infty.$$

This gives the transience we are looking for, so we are done.

□