Math 154: Probability Theory, HW 8

DUE APRIL 2, 2024 BY 9AM

Remember, if you are stuck, take a look at the lemmas/theorems/examples from class, and see if anything looks familiar.

1. SOME PRACTICE WITH MARKOV CHAINS

1.1. **Classification of states.** Consider the state space $\{A, B, C, D\}$. For each Markov chain below (specified by its transition matrix), specify which states (i.e. which of A, B, C, D) are recurrent and which are transient. (Recall a transition matrix P has entries given by $P_{ij} = \mathbb{P}[i \to j]$.)

(1) $P_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(2) $P_2 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

(3) Find two row vectors π_1 and π_2 of length 4 such that $\pi_1 P_1 = \pi_1$ and $\pi_2 P_1 = \pi_2$. Your two row vectors cannot be scalar multiplies of each (e.g. they must be linearly independent). What do you notice about the sign of each entry in π_1, π_2?

1.2. **A nice trick in computing long-time behavior of a Markov chain.** Consider P_1 from Problem 1.1. We will see that diagonalization from linear algebra is actually useful.

(1) Compute $\mathrm{Tr} P_1$ and $\det P_1$.
(2) Compute the eigenvalues of P_1. (*Hint*: the eigenvalues sum to the trace, and they multiply to the determinant. Use part (3) in Problem 1.1.)
(3) Label the eigenvalues as $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$, and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be the associated left eigenvectors, so that $\mathbf{v}_i P_1 = \lambda_i \mathbf{v}_i$. Show that $|\lambda_3|, |\lambda_4| < 1$. Deduce that for $i = 3, 4$, we have $\mathbf{v}_i P_1^n \to \vec{0}$ as $n \to \infty$, where $\vec{0} = (0, 0, 0, 0)$.
(4) Any vector $\mathbf{v}$ can be written as a linear combination $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 + \alpha_4 \mathbf{v}_4$. Show that $\mathbf{v} P_1^n \to \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2$ as $n \to \infty$. This shows that the long-time behavior of the P_1 Markov chain is rather simple!

1.3. **Random walk in dimension** 2**.** Let $\mathbf{X}(n) = (X_1(n), X_2(n))$, where X_1, X_2 are independent symmetric simple random walks such that $X_1(0), X_2(0) = 0$ and $n \geqslant 0$ is an integer.

(1) Show that for any $n \geqslant 0$, we have

$$\mathbb{P}[\mathbf{X}(2n) = (0,0)] = \binom{2n}{n}^2 2^{-4n}.$$

Deduce that $\mathbb{P}[\mathbf{X}(2n) = (0,0)] \geqslant Cn^{-1}$ for all $n \geqslant 1$, where $C \geqslant 0$ is some fixed constant. (*Hint*: use independence of X_1, X_2.)

(2) Show that

$$\sum_{n=1}^{\infty} \mathbb{P}[\mathbf{X}(n) = (0,0)] = \infty.$$

1.4. **Random walk in dimensions greater than or equal to** 3**.** Let $\mathbf{X}(n) = (X_1(n), \ldots, X_d(n))$, where $X_1, \ldots, X_d$ are independent symmetric simple random walks such that $X_1(0), \ldots, X_d(0) = 0$, and $n \geqslant 0$ is an integer and $d \geqslant 3$ is fixed.

(1) Show that $\mathbb{P}[\mathbf{X}(2n) = (0, \ldots, 0)] \leqslant Cn^{-d/2}$ for all $n \geqslant 1$, where C depends only on d.

(2) Show that $\mathbf{X}$ if $d \geqslant 3$, then

$$\sum_{n=1}^{\infty} \mathbb{P}[\mathbf{X}(n) = (0, \ldots, 0)] < \infty.$$

1.5. **Asymmetric simple random walk in dimension** 1**.** Suppose $\{X(n)\}_{n \geqslant 1}$ is a sequence of i.i.d. random variables such that

$$\mathbb{P}[X(n+1) = x] = \begin{cases} p & x = 1 \\ 1-p & x = -1 \\ 0 & \text{else} \end{cases}$$

where $p \neq \frac{1}{2}$. Define $S(n) = X(1) + \ldots + X(n)$ to be the random walk with $S(0) = 0$.

(1) Show that the process $M_n = S(n) - (2p-1)n$ is a martingale with respect to the sequence $\{X(k)\}_{k \geqslant 1}$. Show that $|M_{n+1} - M_n| \leqslant 10$ for all $n \geqslant 0$.

(2) Show that for some constant $C > 0$ independent of $n \geqslant 0$, we have

$$\mathbb{P}\left[|M_n| \geqslant n^{2/3}\right] \leqslant \exp\left\{-Cn^{1/3}\right\}$$

(3) Show that $\mathbb{P}[S(n) = 0] \leqslant \mathbb{P}[|M_n| \geqslant n^{2/3}]$ for n large enough. Using the bound $\exp\{-Cn^{1/3}\} \leqslant C_2 n^{-2}$ for some $C_2 > 0$ fixed, deduce that X has 0 as a transient state. (*Hint*: the assumption $p \neq \frac{1}{2}$ is crucial.)