Math 154: Probability Theory, HW 7

DUE MARCH 19, 2024 BY 9AM

Remember, if you are stuck, take a look at the lemmas/theorems/examples from class, and see if anything looks familiar.

1. GETTING TO KNOW THE CENTRAL LIMIT THEOREM

1.1. **Approximating a complicated expectation.** Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. random variables such that $\mathbb{P}[X_i = \pm 1] = \frac{1}{2}$.

(1) Show that $\mathbb{E}X_i = 0$ and $\mathrm{Var}(X_i) = 1$ for all i.

(2) Define $Y_N := N^{-1/2} \sum_{i=1}^{N} X_i$. Using the central limit theorem, show

$$\lim_{N\to\infty} \mathbb{E}|Y_N| = \int_{\mathbb{R}} |x| \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x.$$

(3) Compute $\lim_{N\to\infty} \mathbb{E}|Y_N|$ by evaluating the integral in part (2).

Solution. (1) Clearly, we have $\mathbb{E}X_i = \frac{1}{2} - \frac{1}{2} = 0$, and $\mathrm{Var}(X_i) = \mathbb{E}X_i^2 = \frac{1}{2} + \frac{1}{2} = 1$.

(2) By (1) and the central limit theorem, we know that $\mathbb{E}|Y_N| \to \mathbb{E}|G|$ with $G \sim N(0,1)$. But this is the RHS of the proposed identity by definition.

(3) We have

$$\int_{\mathbb{R}} |x| \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} x e^{-\frac{x^2}{2}} \mathrm{d}x$$

$$= -\frac{2}{\sqrt{2\pi}} \int_0^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} e^{-\frac{x^2}{2}} \mathrm{d}x = \sqrt{\frac{2}{\pi}}.$$

□

1.2. **Approximating a complicated sum.** Fix any $x \geqslant 0$.

(1) Explain why for any $k \geqslant 0$, we have $2^{-N}\binom{N}{k} = \mathbb{P}[S_N = k]$, where $S_N \sim \mathrm{Bin}(N, \frac{1}{2})$ is a sum of N independent $\mathrm{Bern}(\frac{1}{2})$.

(2) Show that $2S_N - N$ is a sum of N i.i.d. random variables with mean zero and variance 1. Also show that

$$\sum_{k:N^{-1/2}|2k-N|\leqslant x} 2^{-N}\binom{N}{k} = \mathbb{P}\left(-x \leqslant \frac{2S_N - N}{N^{1/2}} \leqslant x\right)$$

(3) Show that as $N \to \infty$, we have

$$\sum_{k:|2k-N|\leqslant xN^{1/2}} 2^{-N}\binom{N}{k} \to \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \mathrm{d}u.$$

(4) (Bonus, +2pt; please do not ask the CAs for help on this one): Show that

$$\sum_{\substack{k:\\ N^{-1/2}|k-N|\leqslant x}} \frac{N^k}{k!} e^{-N} \to_{N\to\infty} \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \mathrm{d}u.$$

(*Hint*: its the same argument; your job is to figure out exactly why.)

Solution. (1) The only way for $S_N = X_1 + \ldots + X_N$ (here, X_i are i.i.d. Bernoulli) to equal $k \geqslant 0$ is for k of the X_i to be 1 and the rest to be 0. In particular, we have $\binom{N}{k}$ many possibilities. Moreover, each has probability 2^{-N}.

(2) We note that $2S_N - N = (2X_1 - 1) + \ldots + (2X_N - 1)$, where X_i are i.i.d. $\mathrm{Bern}(\frac{1}{2})$. Note that $Y_i = 2X_i - 1$ satisfies $\mathbb{E}Y_i = 2\mathbb{E}X_i - 1 = 1 - 1 = 0$ and $\mathrm{Var}(Y_i) = \mathbb{E}Y_i^2 = \mathbb{E}(2X_i - 1)^2 = 4\mathbb{E}X_i^2 - 4\mathbb{E}X_i + 1 = 1$ (since $X_i^2 = X_i$). By part (1), we have

$$\begin{aligned}\sum_{k:N^{-1/2}|2k-N|\leqslant x} 2^{-N}\binom{N}{k} &= \sum_{k:N^{-1/2}|2k-N|\leqslant x} \mathbb{P}\left(S_N = k\right) \\ &= \mathbb{P}\left(N^{-\frac{1}{2}}|2S_N - N| \leqslant x\right) = \mathbb{P}\left(-x \leqslant \frac{2S_N - N}{N^{1/2}} \leqslant x\right),\end{aligned}$$

which finishes the argument.

(3) By part (2), the central limit theorem implies that

$$\mathbb{P}\left(-x \leqslant \frac{2S_N - N}{N^{1/2}} \leqslant x\right) \to \mathbb{P}\left(-x \leqslant G \leqslant x\right) = \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \mathrm{d}u,$$

where $G \sim N(0,1)$. Combine this with part (2).

(4) Note that $e^{-N}N^k/k! = \mathbb{P}(S_N = k)$, where $S_N \sim \mathrm{Pois}(N)$. Moreover, note that S_N has the same distribution as $X_1 + \ldots + X_N$, where X_i are i.i.d. $\mathrm{Pois}(1)$. Using all of

this, we have

$$\sum_{k:N^{-1/2}|k-N|\leqslant x} \frac{N^k}{k!}e^{-N} = \sum_{k:N^{-1/2}|k-N|\leqslant x} \mathbb{P}\left(X_1 + \ldots + X_N = k\right)$$
$$= \mathbb{P}\left(N^{-1/2}|X_1 + \ldots + X_N - N| \leqslant x\right)$$
$$= \mathbb{P}\left(-x \leqslant \frac{(X_1 - 1) + \ldots + (X_N - 1)}{N^{1/2}} \leqslant x\right).$$

Now, note that $\mathbb{E}X_i - 1 = 1 - 1$ and $\mathrm{Var}(X_i - 1) = \mathrm{Var}(X_i) = 1$ if $X_i \sim \mathrm{Pois}(1)$. (Recall that variance is invariant under shift by deterministic constant.) Thus, the central limit theorem shows that the last line converges to $\mathbb{E}G$ with $G \sim N(0,1)$, and we are done.

$\square$

1.3. **Stein's method.** We showed before that if $Z \sim N(0,1)$, then for any smooth function $f : \mathbb{R} \to \mathbb{R}$, we have $\mathbb{E}f'(Z) = \mathbb{E}Zf(Z)$. Conversely, suppose W satisfies the property that for all smooth functions f, we have $\mathbb{E}f'(W) = \mathbb{E}Wf(W)$.

(1) Show that $\mathbb{E}W = 0$ and $\mathbb{E}W^2 = 1$ and $\mathbb{E}W^3 = 0$ and $\mathbb{E}W^4 = 3$.

(2) (Bonus, +2pt; please do not ask the CAs for help on this one): Show that $W \sim N(0,1)$.

Note that this gives a new way of proving the central limit theorem. There are interpretations of this method from physics (in fact, the physicists may argue this is the *right* way to prove the CLT); please see me if you would like to discuss this.

Solution. (1) Take $f(w) = 1$ for all $w \in \mathbb{R}$. We get $0 = \mathbb{E}f'(W) = \mathbb{E}Wf(W) = \mathbb{E}W$. Now, take $f(w) = w$ for all $w \in \mathbb{R}$. We get $1 = \mathbb{E}f'(W) = \mathbb{E}Wf(W) = \mathbb{E}W^2$. Next, take $f(w) = w^2$ for all $w \in \mathbb{R}$. We get $\mathbb{E}f'(W) = 2\mathbb{E}W = 0$ and $\mathbb{E}Wf(W) = \mathbb{E}W^3$, so that $\mathbb{E}W^3 = 0$. Finally, take $f(w) = w^3$. We get $\mathbb{E}f'(W) = 3\mathbb{E}W^2 = 3$ and $\mathbb{E}Wf(W) = \mathbb{E}W^4$, so $\mathbb{E}W^4 = 3$.

(2) We show that $\mathbb{E}W^k = (k-1)!!$ if k is even, and $\mathbb{E}W^k = 0$ if k is odd. This shows that W has the same moments as a Gaussian, and thus it must be a Gaussian random variable. We proceed inductively in k. Suppose k is odd, and $\mathbb{E}W^k = 0$. For $f(w) = w^{k+1}$, we have $\mathbb{E}f'(W) = (k+1)\mathbb{E}W^k = 0$. We also have $\mathbb{E}Wf(W) = \mathbb{E}W^{k+2}$. This shows $\mathbb{E}W^{k+2} = 0$. This finishes the induction for odd k, since the next odd integer after k is $k+2$. Now, suppose k is even. By the induction assumption, for $f(w) = w^{k+1}$, we have $\mathbb{E}f'(W) = (k+1)\mathbb{E}W^k = (k+1)(k-1)!! = (k+1)!!$. We also have $\mathbb{E}Wf(W) = \mathbb{E}W^{k+2}$, so that $\mathbb{E}W^{k+2} = (k+!)!!$. Since the next even integer after k is $k+2$, this completes the induction.

$\square$

1.4. **A little exercise about Fourier transforms.** Suppose $X_N \to X$ and $Y_N \to Y$ in distribution.

(1) Suppose also that X_N, Y_N are independent for each N, and that X, Y are independent. Show that $X_N + Y_N \to X + Y$. (*Hint*: use the Levy continuity theorem)

(2) Give a counterexample to the above when we remove the independence assumptions.

Solution. (1) For any $\xi \in \mathbb{R}$, we have $\mathbb{E}e^{i\xi(X_N+Y_N)} = \mathbb{E}e^{i\xi X_N}e^{i\xi Y_N} = \mathbb{E}e^{i\xi X_N}\mathbb{E}e^{i\xi Y_N}$ since X_N, Y_N are independent. By the assumed weak convergence, we have $\mathbb{E}e^{i\xi X_N}\mathbb{E}e^{i\xi Y_N} \to \mathbb{E}e^{i\xi X}\mathbb{E}e^{i\xi Y} = \mathbb{E}e^{i\xi(X+Y)}$ since X, Y are independent.

(2) Let X_N, Y_N be independent $N(0,1)$. Then $X_N \to X$ and $Y_N \to -X$ in distribution, where $X \sim N(0,1)$. Indeed, note that $-X$ has the same distribution as X if $X \sim N(0,1)$ (the pdf of $N(0,1)$ is an even function). But $X_N + Y_N \sim N(0,2)$ for all N, whereas $X + Y = X - X = 0$ is not $N(0,2)$. So, $X_N + Y_N$ cannot converge in distribution to $X + Y$.

□

1.5. **The moment method.** Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. random variables such that $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 = 1$ and $\mathbb{E}|X_i|^3 < \infty$ for all i. Define $S_N = N^{-1/2}(X_1 + \ldots + X_N)$.

(1) By expanding, show that

$$\mathbb{E}S_N^3 = N^{-\frac{3}{2}} \sum_{i=1}^{N} \mathbb{E}X_i^3 + N^{-\frac{3}{2}} \sum_{1 \leqslant i \neq j \leqslant N} 3\mathbb{E}X_i^2 \mathbb{E}X_j + N^{-\frac{3}{2}} \sum_{i \neq j, j \neq k, i \neq k} \mathbb{E}X_i X_j X_k.$$

(2) Show that $\mathbb{E}S_N^3 \to 0$ as $N \to \infty$.

(3) (Bonus, +1pt; please do not ask the CAs for help on this one): Assume now that $\mathbb{E}|X_i|^4 < \infty$ for all i. Show that $\mathbb{E}S_N^4 \to 3$ by the same type of expansion argument.

Solution. (1) We have

$$S_N^3 = N^{-3/2}(X_1 + \ldots + X_N)^3 = N^{-\frac{3}{2}} \sum_{i,j,k=1}^{N} X_i X_j X_k.$$

Take the case where $i = j = k$; this gives $N^{-3/2} \sum_i X_i^3$. Take the case where exactly two of i, j, k are the same; in this case, we have a term of the form $X_i^2 X_j$ summed over all $i \neq j$, but we also pick up a factor 3 because there are three ways to match exactly two of i, j, k. Finally, take the case where i, j, k are all distinct; this gives $\sum_{i \neq j, j \neq k, i \neq k} X_i X_j X_k$. Thus, we have

$$S_N^3 = N^{-\frac{3}{2}} \sum_{i=1}^{N} X_i^3 + N^{-\frac{3}{2}} \sum_{1 \leqslant i \neq j \leqslant N} 3X_i^2 X_j + N^{-\frac{3}{2}} \sum_{i \neq j, j \neq k, i \neq k} X_i X_j X_k.$$

Now, take expectation (and use linearity of expectation).

(2) Note that $\mathbb{E}X_j = 0$ for all j. Thus, by part (1) and the triangle inequality, we have

$$|\mathbb{E}S_N^3| \leqslant N^{-\frac{3}{2}} \sum_{i=1}^{N} |\mathbb{E}X_i^3| \leqslant N^{-\frac{3}{2}} \sum_{i=1}^{N} \mathbb{E}|X_i|^3 \leqslant CN^{-\frac{1}{2}} \to 0$$

where $C = \mathbb{E}|X_i|^3 < \infty$ (note X_i are i.i.d.).

(3) By expanding as in part (1) and dropping all terms with a factor of $\mathbb{E}X_j = 0$ for some j, we have

$$\mathbb{E}S_N^4 = N^{-2} \sum_{i=1}^{N} \mathbb{E}X_i^4 + 3N^{-2} \sum_{i \neq j} \mathbb{E}X_i^2 \mathbb{E}X_j^2.$$

(Indeed, the number of ways to match each index in $\{i, j, k, \ell\}$ with exactly one other index is 3; it is one of $\{i, j\}$ or $\{i, k\}$ or $\{i, \ell\}$.)By assumption, we know $\mathbb{E}X_i^4 \leqslant C$ for some constant $C < \infty$. Thus, the first term on the RHS is $\leqslant CN^{-1} \to 0$. On the other hand, we have $\mathbb{E}X_i^2 = 1$, so

$$3N^{-2} \sum_{i \neq j} \mathbb{E}X_i^2 \mathbb{E}X_j^2 = 3N^{-2} \sum_{i \neq j} 1 = 3N^{-2} N(N-1) = 3 - 3N^{-1} \to 3,$$

so we are done.

□