Math 154: Probability Theory, HW 7

DUE MARCH 19, 2024 BY 9AM

Remember, if you are stuck, take a look at the lemmas/theorems/examples from class, and see if anything looks familiar.

1. GETTING TO KNOW THE CENTRAL LIMIT THEOREM

1.1. **Approximating a complicated expectation.** Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. random variables such that $\mathbb{P}[X_i = \pm 1] = \frac{1}{2}$.

(1) Show that $\mathbb{E}X_i = 0$ and $\mathrm{Var}(X_i) = 1$ for all i.

(2) Define $Y_N := N^{-1/2} \sum_{i=1}^{N} X_i$. Using the central limit theorem, show

$$\lim_{N\to\infty} \mathbb{E}|Y_N| = \int_{\mathbb{R}} |x| \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x.$$

(3) Compute $\lim_{N\to\infty} \mathbb{E}|Y_N|$ by evaluating the integral in part (2).

1.2. **Approximating a complicated sum.** Fix any $x \geqslant 0$.

(1) Explain why for any $k \geqslant 0$, we have $2^{-N}\binom{N}{k} = \mathbb{P}[S_N = k]$, where $S_N \sim \mathrm{Bin}(N, \frac{1}{2})$ is a sum of N independent $\mathrm{Bern}(\frac{1}{2})$.

(2) Show that $2S_N - N$ is a sum of N i.i.d. random variables with mean zero and variance 1. Also show that

$$\sum_{k: N^{-1/2}|2k-N| \leqslant x} 2^{-N}\binom{N}{k} = \mathbb{P}\left(-x \leqslant \frac{2S_N - N}{N^{1/2}} \leqslant x\right)$$

(3) Show that as $N \to \infty$, we have

$$\sum_{k: |2k-N| \leqslant x N^{1/2}} 2^{-N}\binom{N}{k} \to \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \mathrm{d}u.$$

(4) (Bonus, +2pt; please do not ask the CAs for help on this one): Show that

$$\sum_{\substack{k: \\ N^{-1/2}|k-N| \leqslant x}} \frac{N^k}{k!} e^{-N} \to_{N\to\infty} \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \mathrm{d}u.$$

(*Hint*: its the same argument; your job is to figure out exactly why.)

1.3. **Stein's method.** We showed before that if $Z \sim N(0,1)$, then for any smooth function $f : \mathbb{R} \to \mathbb{R}$, we have $\mathbb{E}f'(Z) = \mathbb{E}Zf(Z)$. Conversely, suppose W satisfies the property that for all smooth functions f, we have $\mathbb{E}f'(W) = \mathbb{E}Wf(W)$.

(1) Show that $\mathbb{E}W = 0$ and $\mathbb{E}W^2 = 1$ and $\mathbb{E}W^3 = 0$ and $\mathbb{E}W^4 = 3$.

(2) (Bonus, +2pt; please do not ask the CAs for help on this one): Show that $W \sim N(0,1)$.

Note that this gives a new way of proving the central limit theorem. There are interpretations of this method from physics (in fact, the physicists may argue this is the *right* way to prove the CLT); please see me if you would like to discuss this.

1.4. **A little exercise about Fourier transforms.** Suppose $X_N \to X$ and $Y_N \to Y$ in distribution.

(1) Suppose also that X_N, Y_N are independent for each N, and that X, Y are independent. Show that $X_N + Y_N \to X + Y$. (*Hint*: use the Levy continuity theorem)

(2) Give a counterexample to the above when we remove the independence assumptions.

1.5. **The moment method.** Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. random variables such that $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 = 1$ and $\mathbb{E}|X_i|^3 < \infty$ for all i. Define $S_N = N^{-1/2}(X_1 + \ldots + X_N)$.

(1) By expanding, show that

$$\mathbb{E}S_N^3 = N^{-\frac{3}{2}} \sum_{i=1}^{N} \mathbb{E}X_i^3 + N^{-\frac{3}{2}} \sum_{1 \leqslant i \neq j \leqslant N} 3\mathbb{E}X_i^2 \mathbb{E}X_j + N^{-\frac{3}{2}} \sum_{i \neq j, j \neq k, i \neq k} \mathbb{E}X_i X_j X_k.$$

(2) Show that $\mathbb{E}S_N^3 \to 0$ as $N \to \infty$.

(3) (Bonus, +1pt; please do not ask the CAs for help on this one): Assume now that $\mathbb{E}|X_i|^4 < \infty$ for all i. Show that $\mathbb{E}S_N^4 \to 3$ by the same type of expansion argument.