# Math 154: Probability Theory, HW 6

DUE MARCH 6, 2024 BY 9AM

*Remember, if you are stuck, take a look at the lemmas/theorems/examples from class, and see if anything looks familiar.*

## 1. TRYING TO PUT EVERYTHING INTO THE LENS OF A MARTINGALE

1.1. **An alternative characterization of conditional expectation.** Take $X_1, \ldots, X_N, Y$ a set of random variables. Let $f : \mathbb{R}^N \to \mathbb{R}$ be any continuous function. Show that

$$\begin{aligned} \mathbb{E}[f(X_1, \ldots, X_N) \cdot Y] &= \mathbb{E}\{\mathbb{E}[f(X_1, \ldots, X_N) \cdot Y | X_1, \ldots, X_N]\} \\ &= \mathbb{E}\{f(X_1, \ldots, X_N)\mathbb{E}[Y | X_1, \ldots, X_N]\}. \end{aligned}$$

It turns out that $\mathbb{E}[Y|X_1, \ldots, X_N]$ is the only random variable which depends only on $X_1, \ldots, X_N$ for which this is true for all continuous $f : \mathbb{R}^N \to \mathbb{R}$. Hence, this is another definition of conditional expectation.

*Solution.* The first identity follows by law of total expectation. For the second, once we condition on $X_1, \ldots, X_N$, the term $f(X_1, \ldots, X_N)$ becomes constant. Then, use linearity of conditional expectation. □

1.2. **Law of large numbers, martingale style.** It turns out independence is not crucial for the law of large numbers to hold, and that a martingale is really the underlying structure in a lot of cases. Let us see why.

Let $(M_N)_{N\geqslant 0}$ be a martingale with respect to the filtration generated by some sequence $(X_n)_{n\geqslant 0}$. We will assume $\sup_{N\geqslant 0}\mathbb{E}|M_{N+1}-M_N|^2<\infty$ and $M_0=0$.

(1) Using $M_N=\sum_{k=0}^{N-1}(M_{k+1}-M_k)$, show that

$$\mathbb{E}|M_N|^2=\sum_{k=0}^{N-1}\mathbb{E}|M_{k+1}-M_k|^2\leqslant CN$$

for some constant $C>0$. (*Hint*: it may help to show that if $j<k$, then

$$\mathbb{E}[(M_{k+1}-M_k)(M_{j+1}-M_j)]=\mathbb{E}\{(M_{j+1}-M_j)\mathbb{E}[M_{k+1}-M_k|X_1,\ldots,X_k]\}=0.$$

To show this, it may help to use Problem 1.1 and the martingale property.)

(2) Show that $\mathbb{P}[|N^{-1}M_N|\geqslant\varepsilon]\leqslant CN^{-1}\varepsilon^{-2}$ for any $\varepsilon>0$ and for some constant $C>0$. (*Hint*: how does one control the tail probability using a second moment?)

(3) Suppose now that $X_n$ are mean $0$ and variance $1$. Define $Y_N=\sum_{n=1}^{N}X_n$ and $Y_0=0$. Show that $\mathbb{P}[|N^{-1}Y_N|\geqslant\varepsilon]\leqslant CN^{-1}\varepsilon^{-2}$ for some constant $C>0$. (This is the law of large numbers as classically stated, e.g. as in class.)

(4) There is no need to get this right or wrong; you will be given credit for any type of guess. Suppose that $\mathbb{E}|M_{N+1}-M_N|^2=1$ for every $N\geqslant 0$. What do you think the distribution of $N^{-1/2}M_N$ converges to as $N\to\infty$? (We never defined what it meant for a distribution to converge, so use an intuitive "definition".)

*Solution.* (1) By expanding and linearity of expectation, we have

$$\mathbb{E}|M_N|^2=\sum_{k=0}^{N-1}\mathbb{E}|M_{k+1}-M_k|^2+2\sum_{j<k}\mathbb{E}[(M_{k+1}-M_k)(M_{j+1}-M_j)].$$

Note that $M_{j+1}-M_j$ is a function of $X_1,\ldots,X_k$ if $k>j$ by definition of a martingale. Thus, we can use Problem 1.1 with $f(X_1,\ldots,X_k)=M_{j+1}-M_j$ to get $\mathbb{E}[(M_{k+1}-M_k)(M_{j+1}-M_j)]=\mathbb{E}\{(M_{j+1}-M_j)\mathbb{E}[M_{k+1}-M_k|X_1,\ldots,X_k]\}$. But this is zero because $\mathbb{E}[M_{k+1}-M_k|X_1,\ldots,X_k]=\mathbb{E}[M_{k+1}|X_1,\ldots,X_k]-M_k=0$ by the martingale property. Thus, the last term on the RHS above vanishes, and thus

$$\mathbb{E}|M_N|^2=\sum_{k=0}^{N-1}\mathbb{E}|M_{k+1}-M_k|^2\leqslant CN,$$

where the bound follows by assumption on the second moments of increments.

(2) By Chebyshev, we have $\mathbb{P}[|N^{-1}M_N|\geqslant\varepsilon]\leqslant\varepsilon^{-2}N^{-2}\mathbb{E}|M_N|^2$. By part (1), we know that $\mathbb{E}|M_N|^2\leqslant CN$ for some constant $C>0$.

(3) Note that $Y_N$ is a martingale with respect to $(X_n)_{n\geqslant 1}$. Indeed, $\mathbb{E}[Y_{N+1}|X_1,\ldots,X_N]=\mathbb{E}[Y_N|X_1,\ldots,X_N]+\mathbb{E}[X_{N+1}|X_1,\ldots,X_N]=Y_N+\mathbb{E}[X_{N+1}]=Y_N$. Now, use part (2).

(4) It "converges" to $N(0,1)$!

□