Math 154: Probability Theory, HW 6

DUE MARCH 5, 2024 BY 9AM

Remember, if you are stuck, take a look at the lemmas/theorems/examples from class, and see if anything looks familiar.

1. TRYING TO PUT EVERYTHING INTO THE LENS OF A MARTINGALE

1.1. **An alternative characterization of conditional expectation.** Take $X_1, \ldots, X_N, Y$ a set of random variables. Let $f : \mathbb{R}^N \to \mathbb{R}$ be any continuous function. Show that

$$\begin{aligned}\mathbb{E}[f(X_1, \ldots, X_N) \cdot Y] &= \mathbb{E}\{\mathbb{E}[f(X_1, \ldots, X_N) \cdot Y | X_1, \ldots, X_N]\} \\ &= \mathbb{E}\{f(X_1, \ldots, X_N)\mathbb{E}[Y | X_1, \ldots, X_N]\}.\end{aligned}$$

It turns out that $\mathbb{E}[Y|X_1, \ldots, X_N]$ is the only random variable which depends only on $X_1, \ldots, X_N$ for which this is true for all continuous $f : \mathbb{R}^N \to \mathbb{R}$. Hence, this is another definition of conditional expectation.

1.2. **Law of large numbers, martingale style.** It turns out independence is not crucial for the law of large numbers to hold, and that a martingale is really the underlying structure in a lot of cases. Let us see why.

Let $(M_N)_{N \geqslant 0}$ be a martingale with respect to the filtration generated by some sequence $(X_n)_{n \geqslant 0}$. We will assume $\sup_{N \geqslant 0} \mathbb{E}|M_{N+1} - M_N|^2 < \infty$ and $M_0 = 0$.

(1) Using $M_N = \sum_{k=0}^{N-1}(M_{k+1} - M_k)$, show that

$$\mathbb{E}|M_N|^2 = \sum_{k=0}^{N-1} \mathbb{E}|M_{k+1} - M_k|^2 \leqslant CN$$

for some constant $C > 0$. (*Hint*: it may help to show that if $j < k$, then

$$\mathbb{E}[(M_{k+1} - M_k)(M_{j+1} - M_j)] = \mathbb{E}\{(M_{j+1} - M_j)\mathbb{E}[M_{k+1} - M_k | X_1, \ldots, X_k]\} = 0.$$

To show this, it may help to use Problem 1.1 and the martingale property.)

(2) Show that $\mathbb{P}[|N^{-1}M_N| \geqslant \varepsilon] \leqslant CN^{-1}\varepsilon^{-2}$ for any $\varepsilon > 0$ and for some constant $C > 0$. (*Hint*: how does one control the tail probability using a second moment?)

(3) Suppose now that X_n are mean 0 and variance 1 and jointly independent. Define $Y_N = \sum_{n=1}^{N} X_n$ and $Y_0 = 0$. Show that $\mathbb{P}[|N^{-1}Y_N| \geqslant \varepsilon] \leqslant CN^{-1}\varepsilon^{-2}$ for some constant $C > 0$. (This is the law of large numbers as classically stated, e.g. as in class.)

(4) There is no need to get this right or wrong; you will be given credit for any type of guess. Suppose that $\mathbb{E}|M_{N+1} - M_N|^2 = 1$ for every $N \geqslant 0$. What do you think the distribution of $N^{-1/2}M_N$ converges to as $N \to \infty$? (We never defined what it meant for a distribution to converge, so use an intuitive "definition".)