Math 154: Probability Theory, HW 5

DUE MARCH 6, 2024 BY 9AM

Remember, if you are stuck, take a look at the lemmas/theorems/examples from class, and see if anything looks familiar.

1. SOME PRACTICE WITH MARTINGALES

1.1. **Polya's urn.** This is perhaps the most important urn model in probability. An urn contains r red and g green balls, where $r, g > 0$. A ball is drawn from the urn, its color is noted, it is returned to the urn, and another ball of the same color is also added to the urn. Let R_n denote the number of red balls drawn after n draws.

(1) Suppose $r = 1$. Show that $Y_n = \frac{1+R_n}{n+r+g}$ for $n \geqslant 0$ is a martingale with respect to the filtration generated by $(R_n)_{n\geqslant 0}$, and show that $\sup_{n\geqslant 1} |Y_n| \leqslant C$ for some constant $C > 0$.

(2) Suppose $r, g = 1$. Let T be the number of turns that is needed to draw a green ball. Show that $\mathbb{E}\frac{1}{T+2} = \frac{1}{4}$. (Justify the application of any theorem you may be using!)

Solution. (1) To prove $\sup_n |Y_n| \leqslant C$, it suffices to note that $R_n \leqslant n$ for all n, since we can draw at most n red balls in n steps. In particular, this shows $|Y_n| \leqslant \frac{1+n}{n+r+g} \leqslant 1$, since $r + g \geqslant 1$. For the martingale property, we have to check that for any $n \geqslant 1$, we have $\mathbb{E}[Y_{n+1}|R_1, \ldots, R_n] = Y_n$. By definition, we have

$$\begin{aligned} Y_{n+1} &= \frac{1+R_{n+1}}{n+1+r+g} = \frac{1+R_n+X_{n+1}}{n+1+r+g} \\ &= \frac{1+R_n}{n+r+g} - (1+R_n)\left(\frac{1}{n+1+r+g} - \frac{1}{n+r+g}\right) + \frac{X_{n+1}}{n+1+r+g} \\ &= Y_n + \frac{1+R_n}{(n+1+r+g)(n+r+g)} + \frac{X_{n+1}}{n+1+r+g}. \end{aligned}$$

where X_{n+1} is 1 is a red ball is drawn at step $n+1$ and 0 otherwise. If we condition on $R_1, \ldots, R_n$, then we know that there are $r + R_n$ red balls and $g + n - R_N$ green balls in the urn at time n before we draw for the $(n+1)$-st time. Thus, the probability of $X_{n+1} = 1$ after this conditioning is equal to $\frac{r+R_n}{n+r+g}$, and

$$\mathbb{E}\left[\frac{X_{n+1}}{n+1+r+g}\middle| R_1, \ldots, R_n\right] = \frac{r+R_n}{(n+r+g)(n+1+r+g)}.$$

We now note that if we condition on $R_1, \ldots, R_n$, then Y_n and R_n are fixed. By combining this with the previous two displays, we have

$$\mathbb{E}[Y_{n+1}|R_1, \ldots, R_n] = Y_n - \frac{1+R_n}{(n+1+r+g)(n+r+g)} - \frac{r+R_n}{(n+1+r+g)(n+r+g)}.$$

Now use the assumption $r = 1$ to conclude.

(2) Note that T is a stopping time with respect to the filtration of $\{R_n\}_{n\geqslant 0}$, since the $\mathbf{1}_{T\leqslant n}$ is constant/deterministic once we condition on $R_1, \ldots, R_n$ for any $n \geqslant 0$. Moreover, we know that Y_n is a uniformly bounded martingale by part (1). Thus, we know that $\mathbb{E}[Y_T] = \mathbb{E}[Y_0]$. By definition, this gives

$$\mathbb{E}\left[\frac{1+R_T}{T+2}\right] = \frac{1}{2}.$$

Since T is the number of turns needed to draw a green ball, we know that $R_T = T-1$. Thus, the previous line becomes $\mathbb{E}\frac{T}{T+2} = 1 - \mathbb{E}\frac{2}{T+2} = \frac{1}{2}$, at which point the claim is immediate.

□

1.2. **Bernstein's inequality.** Suppose $X_1, \ldots, X_i, \ldots \sim \mathrm{Bern}(p)$ are i.i.d., and define $Y_i = X_i - p$ for $i = 1, \ldots, N$. Prove that there exists a constant $C > 0$ such that for any $\varepsilon > 0$, we have

$$\mathbb{P}\left[\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N} Y_i\right| \geqslant \varepsilon\right] \leqslant \exp\left[-C\varepsilon^2\right].$$

In particular, even though the maximum value of $Y_1 + \ldots + Y_N$ can grow linearly in N, it likes to stay around $\sqrt{N}$. (*Hint*: the process $S_N = Y_1 + \ldots + Y_N$ is a martingale with respect to the filtration generated by $(X_n)_{n \geqslant 1}$; check this!)

Solution. Per the hint, let us check that $S_N = Y_1 + \ldots + Y_N$ is a martingale with respect to the proposed filtration. We must check that $\mathbb{E}[S_{N+1}|X_1, \ldots, X_N] = S_N$. To this end, we write $S_{N+1} = S_N + Y_{N+1}$. If we condition on $X_1, \ldots, X_N$, then $Y_1, \ldots, Y_N$ are fixed, and thus so is S_N. Thus, we have $\mathbb{E}[S_{N+1}|X_1, \ldots, X_N] = S_N + \mathbb{E}[Y_{N+1}|X_1, \ldots, X_N]$. But Y_{N+1} is independent of $X_1, \ldots, X_N$ and mean 0, so $\mathbb{E}[Y_{N+1}|X_1, \ldots, X_N] = \mathbb{E}Y_{N+1} = 0$. Now, it suffices to use the Azuma inequality, since $Y_1, \ldots$ are uniformly bounded in N with probability 1. $\square$

1.3. **Maximal version of Bernstein's inequality.** We have shown that the running sum of independent Bernoulli's has "sub-Gaussian behavior" in Problem 1.2. We will show something similar but for the "maximal process".

Recall notation from Problem 1.2. Define $X_N := N^{-\frac{1}{2}} \sup_{1 \leqslant n \leqslant N} |Y_1 + \ldots + Y_n|$.

(1) Show that for any $p \geqslant 2$, we have $\mathbb{E}|X_N|^p \leqslant \left(\frac{p}{p-1}\right)^p \mathbb{E}|N^{-\frac{1}{2}} \sum_{i=1}^N Y_i|^p$.

(2) Use Problem 1.2 and the previous part to show that for some constant $C > 0$, we have

$$\mathbb{E}|X_N|^{2p} \leqslant \left(\frac{2p}{2p-1}\right)^{2p} (2p-1)!!C^p$$

for any integer $p \geqslant 1$.

(3) Use the previous part to show that there exists a constant $K > 0$ such that for any $\varepsilon > 0$, we have

$$\mathbb{P}\left[|X_N| \geqslant \varepsilon\right] \leqslant \exp[-K\varepsilon^2].$$

Solution. (1) We showed already that $S_N = Y_1 + \ldots + Y_N$ is a martingale with respect to the filtration generated by $X_1, \ldots$, so that it suffices to just use Doob's maximal inequality.

(2) By Problem 1.2, and the equivalence of Gaussian moments and Gaussian tail probabilities, we know that $\mathbb{E}|N^{-1/2} \sum_{i=1}^N Y_i|^p \leqslant (2p-1)!!C^p$ for some constatnt C. Now, combine this with the part (1).

(3) We know that $\frac{2p}{2p-1} \leqslant L$ for all p and for some constant $L > 0$. Thus, part (2) implies that $\mathbb{E}|X_N|^{2p} \leqslant (2p-1)!!(CL)^p$. Now use equivalence of Gaussian moments and Gaussian tail probabilities.

□

1.4. **Gambler's ruin for an unfair game.** Let $\{X_n\}_{n\geqslant 1}$ be independent $\mathrm{Bern}(p)$ random variables with $p \neq 0, \frac{1}{2}, 1$. Define $S_N = S_{N-1} + (-1)^{1+X_N}$ for $N \geqslant 1$ and set $S_0 = 0$.

(1) Show that $M_N = \left(\frac{1-p}{p}\right)^{S_N}$ is a martingale with respect to the filtration generated by $(X_n)_{n\geqslant 1}$.
(2) Let τ be the first positive integer such that $S_\tau = -a$ or $S_\tau = b$ for $a, b > 0$ fixed. Compute $\mathbb{P}[S_\tau = -a]$ in terms of a, b, p.

Solution. (1) We have

$$\begin{aligned}\mathbb{E}[M_{N+1}|X_1,\ldots,X_N] &= \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{S_{N+1}}\middle| X_1,\ldots,X_N\right]\\ &= \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{S_N}\left(\frac{1-p}{p}\right)^{(-1)^{1+X_{N+1}}}\middle| X_1,\ldots,X_N\right]\\ &= \left(\frac{1-p}{p}\right)^{S_N}\mathbb{E}\left[\left(\frac{1-p}{p}\right)^{(-1)^{1+X_{N+1}}}\middle| X_1,\ldots,X_N\right].\end{aligned}$$

Note the first factor in the last line is just M_N. Since X_i are jointly independent, the expectation in the last line is just $\mathbb{E}[(\frac{1-p}{p})^{(-1)^{1+X_{N+1}}}] = \frac{p}{1-p}(1-p) + \frac{1-p}{p}p = 1$. We conclude that $\mathbb{E}[M_{N+1}|X_1,\ldots,X_N] = M_N$, so the martingale property holds.

(2) Note that τ is a stopping time with respect to the filtration generated by $X_1, \ldots$, since $\mathbf{1}_{\tau\leqslant n}$ is deterministic once we condition on $X_1,\ldots,X_n$ for any $n \geqslant 0$. Moreover, note that $|M_n|$ is uniformly bounded for all $n \leqslant \tau$; indeed, for $n \leqslant \tau$, we know that S_n is uniformly bounded. Thus, by the optional stopping theorem, we have

$$1 = \mathbb{E}[M_0] = \mathbb{E}[M_\tau] = \left(\frac{1-p}{p}\right)^{-a}\mathbb{P}[S_\tau = -a] + \left(\frac{1-p}{p}\right)^{b}\mathbb{P}[S_\tau = b].$$

Note that $\mathbb{P}[S_\tau = b] = 1 - \mathbb{P}[S_\tau = -a]$. Thus, the previous display becomes

$$\mathbb{P}[S_\tau = -a]\left\{\left(\frac{1-p}{p}\right)^{-a} - \left(\frac{1-p}{p}\right)^{b}\right\} = 1 - \left(\frac{1-p}{p}\right)^{b}.$$

In particular, we have

$$\mathbb{P}[S_\tau = -a] = \frac{1-\left(\frac{1-p}{p}\right)^{b}}{\left(\frac{1-p}{p}\right)^{-a} - \left(\frac{1-p}{p}\right)^{b}}$$

□

1.5. **The "quadratic" process of a martingale, and the Ito martingale.**

(1) Suppose that $\{X_n\}_{n\geqslant 1}$ are independent mean zero random variables with variances $\sigma_i^2 = \mathbb{E}X_i^2$. Show that $Y_N := \sum_{i=1}^N X_i^2 - \sum_{i=1}^N \sigma_i^2$ with $Y_0 = 0$ is a martingale with respect to the filtration generated by $\{X_n\}_{n\geqslant 1}$.

(2) Suppose in addition that X_i are i.i.d. $\mathrm{Bern}(\frac{1}{2})$, and define $W_i = (-1)^{1+X_i}$. For any function $f : \mathbb{Z} \to \mathbb{R}$, define its *Laplacian* to be $\Delta f(x) = f(x+1)+f(x-1)-2f(x)$. Moreover, define $Z_N = W_1 + \ldots + W_N$. Show that $f(Z_N) - \sum_{i=1}^N \frac{1}{2}\Delta f(Z_i)$ is a martingale with respect to the filtration generated by $\{X_n\}_{n\geqslant 1}$.

Solution. (1) We have $\mathbb{E}[Y_{N+1}|X_1,\ldots,X_N] = \mathbb{E}[Y_N|X_1,\ldots,X_N]+\mathbb{E}[X_{N+1}^2|X_1,\ldots,X_N]-\sigma_{N+1}^2$. The first term is just Y_N since Y_N is a function of just $X_1,\ldots,X_N$. The second term is just $\mathbb{E}[X_{N+1}^2] - \sigma_{N+1}^2 = 0$ since X_i are independent.

(2) We have

$$\mathbb{E}\left[f(Z_{N+1}) - \sum_{i=1}^N \frac{1}{2}\Delta f(Z_i)\middle| X_1,\ldots,X_N\right]$$

$$= \mathbb{E}\left[f(Z_{N+1})|X_1,\ldots,X_N\right] - \sum_{i=1}^N \frac{1}{2}\Delta f(Z_i)$$

since $Z_1,\ldots,Z_N$ are fixed after we condition on $X_1,\ldots,X_N$. Now, note that $Z_{N+1} = Z_N + W_{N+1}$, and W_{N+1} is ± 1-valued with equal probabilities and independent of $X_1,\ldots,X_N$. Thus, we have $\mathbb{E}[f(Z_{N+1})|X_1,\ldots,X_N] = \frac{1}{2}f(Z_N+1) + \frac{1}{2}f(Z_N-1)$. In particular, we have $\mathbb{E}[f(Z_{N+1})|X_1,\ldots,X_N] - \frac{1}{2}\Delta f(Z_N) = f(Z_N)$, and we get

$$\mathbb{E}\left[f(Z_{N+1}) - \sum_{i=1}^N \frac{1}{2}\Delta f(Z_i)\middle| X_1,\ldots,X_N\right]$$

$$= f(Z_N) - \sum_{i=1}^{N-1} \frac{1}{2}\Delta f(Z_i).$$

□

1.6. **Gaussian tail probabilities implies Gaussian moments.** Suppose X is a continuous random variable such that $\mathbb{P}[|X| \geqslant C] \leqslant \exp\{-KC^2\}$ for all $C > 0$ (K is just a fixed constant).

(1) Let p be the pdf of X and fix $q \geqslant 1$. Justify each line in the following:

$$\begin{aligned}\int_{\mathbb{R}} x^{2q}p(x)dx &= 2q\int_0^\infty x^{2q}p(x)dx + 2q\int_0^\infty x^{2q}p(-x)dx \\ &= 2q\int_0^\infty x^{2q-1}\left(\int_x^\infty p(u)du\right)dx + 2q\int_0^\infty x^{2q-1}\left(\int_x^\infty p(-u)du\right)dx \\ &\leqslant 4q\int_0^\infty x^{2q-1}\mathbb{P}[|X| \geqslant x]dx \\ &\leqslant 4q\int_0^\infty x^{2q-1}\exp\{-Kx^2\}dx.\end{aligned}$$

(*Hint*: integration-by-parts is your friend.)

(2) Show that $\mathbb{E}|X|^{2q} \leqslant 4qK^{-q}\int_0^\infty y^{2q-1}\exp\{-y^2\}dy$.

(3) (Bonus, +2pt): Show that $\int_0^\infty y^{2q-1}\exp\{-y^2\}dy \leqslant C_1(2q-1)!!C_2^q$ for some constants $C_1, C_2 > 0$.

Solution. (1) The first line follows by decomposing the integral over $\mathbb{R}$ into integrals over $(-\infty, 0]$ and $[0, \infty)$. The second line follows by integrating-by-parts; indeed,

$$\begin{aligned}\int_0^\infty x^{2q}p(x)dx &= -\int_0^\infty x^{2q}\left(\frac{d}{dx}\int_x^\infty p(u)du\right)dx \\ &= \left(-x^{2q}\int_x^\infty p(u)du\right)\Big|_{x=0}^{x=\infty} + \int_0^\infty 2qx^{2q-1}\int_x^\infty p(u)dudx.\end{aligned}$$

The first term is zero, since $x^{2q}|_{x=0}$ is 0, and $\int_x^\infty p(u)du \leqslant \exp\{-Kx^2\}$ goes to zero faster than x^{2q} goes to infinity as $x \to \infty$. For the second integration in the first line, the same argument works. To justify the third line, we just note that $\int_x^\infty p(u)du = \mathbb{P}[X \geqslant x] \leqslant \mathbb{P}[|X| \geqslant x]$ and $\int_x^\infty p(-u)du = \mathbb{P}[X \leqslant -x] \leqslant \mathbb{P}[|X| \geqslant x]$. The last line holds by assumption.

(2) Using the u-substitution $K^{1/2}x = u$, so that $dx = K^{-1/2}du$, we have

$$\begin{aligned}\int_0^\infty x^{2q-1}\exp\{-Kx^2\}dx &= K^{-\frac{2q-1}{2}}\int_0^\infty (K^{\frac{1}{2}}x)^{2q-1}\exp\{-Kx^2\}dx \\ &= K^{-q}\int_0^\infty y^{2q-1}\exp\{-y^2\}dy.\end{aligned}$$

□