Math 154: Probability Theory, HW 5

DUE MARCH 5, 2024 BY 9AM

Remember, if you are stuck, take a look at the lemmas/theorems/examples from class, and see if anything looks familiar.

1. SOME PRACTICE WITH MARTINGALES

1.1. **Polya's urn.** This is perhaps the most important urn model in probability. An urn contains r red and g green balls, where $r, g > 0$. A ball is drawn from the urn, its color is noted, it is returned to the urn, and another ball of the same color is also added to the urn. Let R_n denote the number of red balls drawn after n draws.

(1) Suppose $r = 1$. Show that $Y_n = \frac{1+R_n}{n+r+g}$ for $n \geqslant 0$ is a martingale with respect to the filtration generated by $(R_n)_{n\geqslant 1}$, and show that $\sup_{n\geqslant 1} |Y_n| \leqslant C$ for some constant $C > 0$.
(2) Suppose $r, g = 1$. Let T be the number of turns that is needed to draw a green ball. Show that $\mathbb{E}\frac{1}{T+2} = \frac{1}{4}$. (Justify the application of any theorem you may be using!)

1.2. **Bernstein's inequality.** Suppose $X_1, \ldots \sim \mathrm{Bern}(p)$ are i.i.d., and define $Y_i = X_i - p$ for $i = 1, \ldots, N$. Prove that there exists a constant $C > 0$ such that for any $\varepsilon > 0$, we have

$$\mathbb{P}\left[\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N} Y_i\right| \geqslant \varepsilon\right] \leqslant \exp\left[-C\varepsilon^2\right].$$

In particular, even though the maximum value of $Y_1 + \ldots + Y_N$ can grow linearly in N, it likes to stay around $\sqrt{N}$. (***Hint***: the process $S_N = Y_1 + \ldots + Y_N$ is a martingale with respect to the filtration generated by $(X_n)_{n\geqslant 1}$; check this!)

1.3. **Maximal version of Bernstein's inequality.** We have shown that the running sum of independent Bernoulli's has "sub-Gaussian behavior" in Problem 1.2. We will show something similar but for the "maximal process".

Recall notation from Problem 1.2. Define $X_N := N^{-\frac{1}{2}} \sup_{1\leqslant n\leqslant N} |Y_1 + \ldots + Y_n|$.

(1) Show that for any $p \geqslant 2$, we have $\mathbb{E}|X_N|^p \leqslant \left(\frac{p}{p-1}\right)^p \mathbb{E}|N^{-\frac{1}{2}} \sum_{i=1}^{N} Y_i|^p$.
(2) Use Problem 1.2 and the previous part to show that for some constant $C > 0$, we have

$$\mathbb{E}|X_N|^{2p} \leqslant \left(\frac{2p}{2p-1}\right)^{2p} (2p-1)!!C^p$$

for any integer $p \geqslant 1$.

(3) Use the previous part to show that there exists a constant $K > 0$ such that for any $\varepsilon > 0$, we have

$$\mathbb{P}\left[|X_N| \geqslant \varepsilon\right] \leqslant \exp[-K\varepsilon^2].$$

(*Hint*: see the end of the notes for week 5 for a helpful lemma.)

1.4. **Gambler's ruin for an unfair game.** Let $\{X_n\}_{n\geqslant 1}$ be independent $\mathrm{Bern}(p)$ random variables with $p \neq 0, \frac{1}{2}, 1$. Define $S_N = S_{N-1} + (-1)^{1+X_N}$ for $N \geqslant 1$ and set $S_0 = 0$.

(1) Show that $M_N = \left(\frac{1-p}{p}\right)^{S_N}$ is a martingale with respect to the filtration generated by $(X_n)_{n\geqslant 1}$.
(2) Let τ be the first positive integer such that $S_\tau = -a$ or $S_\tau = b$ for $a, b > 0$ fixed. Compute $\mathbb{P}[S_\tau = -a]$ in terms of a, b, p.

1.5. **The "quadratic" process of a martingale, and the Ito martingale.**

(1) Suppose that $\{X_n\}_{n\geqslant 1}$ are independent mean zero random variables with variances $\sigma_i^2 = \mathbb{E}X_i^2$. Show that $Y_N := \sum_{i=1}^N X_i^2 - \sum_{i=1}^N \sigma_i^2$ with $Y_0 = 0$ is a martingale with respect to the filtration generated by $\{X_n\}_{n\geqslant 1}$.
(2) Suppose in addition that X_i are i.i.d. $\mathrm{Bern}(\frac{1}{2})$, and define $W_i = (-1)^{1+X_i}$. For any function $f : \mathbb{Z} \to \mathbb{R}$, define its *Laplacian* to be $\Delta f(x) = f(x+1) + f(x-1) - 2f(x)$. Moreover, define $Z_N = W_1 + \ldots + W_N$. Show that $f(Z_N) - \sum_{i=1}^{N-1} \frac{1}{2}\Delta f(Z_i)$ is a martingale with respect to the filtration generated by $\{X_n\}_{n\geqslant 1}$.

1.6. **Gaussian tail probabilities implies Gaussian moments.** Suppose X is a continuous random variable such that $\mathbb{P}[|X| \geqslant C] \leqslant \exp\{-KC^2\}$ for all $C > 0$ (K is just a fixed constant). We showed in class that $\mathbb{E}|X|^{2q} \leqslant C_1(2q-1)!!C_2^q$ for all $q \geqslant 1$ and for some $C_1, C_2 > 0$ implies $\mathbb{P}[|X| \geqslant C] \leqslant \exp\{-KC^2\}$ for some $K > 0$. We now show the converse is true.

(1) Let p be the pdf of X. Show

$$\begin{aligned}
\int_{\mathbb{R}} x^{2q} p(x)dx &= \int_0^\infty x^{2q} p(x)dx + \int_0^\infty x^{2q} p(-x)dx \\
&= 2q \int_0^\infty x^{2q-1} \left(\int_x^\infty p(u)du\right) dx + 2q \int_0^\infty x^{2q-1} \left(\int_x^\infty p(-u)du\right) dx \\
&\leqslant 4q \int_0^\infty x^{2q-1} \mathbb{P}[|X| \geqslant x]dx \\
&\leqslant 4q \int_0^\infty x^{2q-1} \exp\{-Kx^2\}dx.
\end{aligned}$$

(*Hint*: integration-by-parts is your friend.)
(2) Using u-substitution, show that $\mathbb{E}|X|^{2q} \leqslant 4qK^{-q} \int_0^\infty y^{2q-1} \exp\{-y^2\}dy$.
(3) (Bonus, +2pt): Show that $\int_0^\infty y^{2q-1} \exp\{-y^2\}dy \leqslant C_1(2q-1)!!C_2^q$ for some constants $C_1, C_2 > 0$.