Math 154: Probability Theory, HW 4

DUE FEB 20, 2024 BY 9AM

Remember, if you are stuck, take a look at the lemmas/theorems/examples from class, and see if anything looks familiar.

1. TIME TO GET TO COMPUTATIONS

1.1. Laplace transform of an exponential random variable. Let $X \sim \mathrm{Exp}(\lambda)$ (for $\lambda > 0$).

(1) Show that $\mathbb{E}e^{\xi X} = \frac{\lambda}{\lambda-\xi}$ for all $0 \leqslant \xi < \lambda$, so that $\mathbb{E}e^{\xi X} < \infty$ if and only if $\xi < \lambda$ (you don't need to prove this last claim).
(2) Compute $\mathbb{E}X^k$ for $k = 0, 1, 2, 3, 4$.
(3) Show that for any $\xi \in \mathbb{R}$, we have $\mathbb{E}e^{i\xi X} = \frac{\lambda}{\lambda - i\xi}$ for all $\xi \in \mathbb{R}$.

1.2. Laplace transform of a Poisson random variable. Let $X \sim \mathrm{Pois}(\lambda)$.

(1) Show that $\mathbb{E}e^{\xi X} = e^{\lambda(e^{\xi}-1)}$.
(2) Compute $\mathbb{E}X^k$ for $k = 1, 2, 3$.
(3) Use part (1) to show that if $X \sim \mathrm{Pois}(\lambda)$ and $Y \sim \mathrm{Pois}(\mu)$ are independent, then $X + Y \sim \mathrm{Pois}(\lambda + \mu)$.

1.3. Cauchy distribution. We say that $X \sim \mathrm{Cauchy}$ if it is a continuous random variable on $\mathbb{R}$ with pdf $p(x) = \frac{1}{\pi(1+x^2)}$.

(1) Show that $\int_{\mathbb{R}} p(x)dx = 1$ using calculus, so that $p(x)$ is actually a pdf. (You can look up the antiderivative of $\frac{1}{1+x^2}$ and its properties; this is more just a check for you to do.)
(2) Show that $\mathbb{E}|X| = \infty$.
(3) Show that for any $\xi \in \mathbb{R}$, we have
$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{-|\xi|}e^{-ix\xi}d\xi = \frac{1}{\pi(1+x^2)}.$$
Conclude that if $X \sim \mathrm{Cauchy}$, then $\mathbb{E}e^{i\xi X} = e^{-|\xi|}$. Can you briefly explain briefly why this formula alone suggests that $\mathbb{E}X$ is not well-defined?

1.4. A concentration inequality. Suppose $X_1, \ldots, X_N$ are i.i.d. random variables (i.e. they are independent and have the same distribution), and suppose $\mathbb{E}X_i = 0$ and $\mathbb{E}e^{\lambda X_i} < \infty$ for all $\lambda \in \mathbb{R}$. Let $Y = \frac{X_1 + \ldots + X_N}{N}$.

(1) Compute $\mathbb{E}e^{\lambda Y}$ in terms of the moment generating functions of $X_1, \ldots, X_N$.

(2) Show that for any constants $\lambda, c > 0$,

$$\mathbb{P}[|Y| \geqslant c] \leqslant e^{-c\lambda}\mathbb{E}e^{\lambda Y} + e^{-c\lambda}\mathbb{E}e^{-\lambda Y} = e^{-c\lambda}\left(\prod_{i=1}^{N}\mathbb{E}e^{\frac{\lambda X_i}{N}} + \prod_{i=1}^{N}\mathbb{E}e^{\frac{-\lambda X_i}{N}}\right).$$

(*Hint*: the LHS is $\leqslant \mathbb{P}[Y \geqslant c] + \mathbb{P}[-Y \geqslant c]$.)

(3) Using the inequality $e^x \leqslant 1 + x + x^2 e^x$, show that $\mathbb{E}e^{\frac{\lambda X_i}{N}} \leqslant 1 + \frac{\lambda^2}{N^2}\mathbb{E}[X_i^2 e^{\frac{\lambda X_i}{N}}]$.

(4) We will now choose $\lambda = N^{1/2}$. Using the inequality $x^2 e^{\kappa x} \leqslant e^{2x} + e^{-2x}$ for any $x \in \mathbb{R}$ and any $|\kappa| \leqslant 1$, show that $\mathbb{E}X_i^2 e^{\frac{\lambda X_i}{N}} \leqslant \mathbb{E}e^{2X_i} + \mathbb{E}e^{-2X_i}$, and thus $\mathbb{E}e^{\frac{\lambda X_i}{N}} \leqslant 1 + \frac{C}{N}$ for some constant C.

(5) You can take for granted that the same argument shows $\mathbb{E}e^{-\frac{\lambda X_i}{N}} \leqslant 1 + \frac{C}{N}$. Using the inequality $(1 + \frac{C}{N})^N \leqslant e^C$, show that $\mathbb{P}[|Y| \geqslant c] \leqslant 2e^{-c\sqrt{N}}e^C$.

1.5. **An application of the law of large numbers.** Suppose I give you a coin and tell you that the probability of heads is 0.48. Suppose you want to test if I am right. How many times N do you have to flip this coin to be at least 95% confident that it is biased towards heads? To be precise:

(1) Let $X_1, \ldots, X_N \sim \mathrm{Bern}(p)$ with $p = 0.48$ be independent. Set $Y = \frac{1}{N}\sum_{i=1}^{N} X_i$. Recall $\mathbb{E}Y = p$. Using the bound

$$\mathbb{P}[|Y - p| \geqslant 0.02] \leqslant \frac{\mathrm{Var}(X_1)}{N(0.02)^2}$$

from class, how large do you have to take N for this probability to be $\leqslant 5\%$?

(2) What if we instead use the following bound (which is what you get when optimizing in Problem 1.4):

$$\mathbb{P}[|Y - p| \geqslant 0.02] \leqslant 2e^{-0.02\sqrt{N}}\mathbb{E}e^{X_1}.$$

Which bound produces the smaller N?