Math 154: Probability Theory, HW 3

DUE FEB 3, 2024 BY 9AM

Remember, if you are stuck, take a look at the lemmas/theorems/examples from class, and see if anything looks familiar.

1. ALL OF THESE PROBLEMS REQUIRE AT LEAST A LITTLE THOUGHT

1.1. **Some magic in the Gaussian.** Suppose $X \sim N(0,1)$.

(1) Show that

$$xe^{-\frac{x^2}{2}} = -\frac{d}{dx}e^{-\frac{x^2}{2}}$$

(2) Take any smooth function $f : \mathbb{R} \to \mathbb{R}$. Show that

$$\mathbb{E}Xf(X) = \mathbb{E}f'(X),$$

provided that both sides converge absolutely (when written as integrals). This is often known as *Gaussian integration by parts*. (*Hint*: the hint is in the name.)

(3) Show that for any integer $k \geqslant 0$, we have $\mathbb{E}X^{2k+1} = 0$.

(4) Show that for any integer $k \geqslant 0$, we have $\mathbb{E}X^{2k} = (2k-1)!!$, where $(2k-1)!! := (2k-1)(2k-3)\ldots 1$. (*Hint*: use part (2) with $f(X) = X^{2k-1}$, and induct on k.)

1.2. **Another fact about the Gaussian distribution.** Let $X \sim N(0,\sigma^2)$ for some $\sigma > 0$. Take any $\lambda \in \mathbb{R}$. Show that

$$\mathbb{E}e^{\lambda X} = e^{\frac{\lambda^2\sigma^2}{2}}.$$

(*Hint*: you may want to use the completing-the-square formula $a^2 - 2ba = (a-b)^2 - b^2$ after you write out what the expectation on the LHS is as an integral on $\mathbb{R}$.) Give another proof of $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = \sigma^2$ by differentiating both sides of this identity (once and twice) and setting $\lambda = 0$.

1.3. **How does one sample from a distribution?** Suppose X is a continuous random variable, so that $\mathbb{P}(X \leqslant x) = \int_{-\infty}^{x} p(u)du$. Suppose p is smooth and $p(u) > 0$ for all $u \in \mathbb{R}$.

(1) Show that the distribution of the random variable

$$F(X) = \int_{-\infty}^{X} p(u)du$$

is the uniform distribution on $[0,1]$. (Here, we evaluate the top limit of the integral at the random variable X. *Hint*: it is not important to know what its inverse exactly is.)

(2) Show that the random variable $-\log F(X)$ has p.d.f given by e^{-x}.

1.4. **What?** Suppose X is an exponential random variable (i.e. it has the exponential distribution). Show that $\mathbb{P}(X > s+x | X > s) = \mathbb{P}(X > x)$ for any $x, s \geqslant 0$.

1.5. **To the right or to the left?** Let X have variance σ^2, and write $m_k = \mathbb{E}X^k$. Define the *skewness* of (the distribution of) X to be $\mathrm{skw}(X) = \frac{\mathbb{E}(X-m_1)^3}{\sigma^3}$. (This measures how much to the left/right the graph of the pdf is.)

(1) Show that $\mathrm{skw}(X) = \frac{m_3 - 3m_1m_2 + 2m_1^3}{\sigma^3}$

(2) Let $X_1, \ldots, X_n$ be i.i.d. copies of X (i.e. they are independent and have the same distribution). Set $S_n = X_1 + \ldots + X_n$. Using the following, show $\mathrm{skw}(S_n) = \frac{\mathrm{skw}(X_1)}{\sqrt{n}}$.
- Compute $\mathrm{Var}(S_n)$ in terms of $\mathrm{Var}(X_1)$ using the i.i.d. property of $X_1, \ldots, X_n$.
- Show that $\mathbb{E}S_n = n\mathbb{E}X_1$.
- Letting $m = \mathbb{E}X_1$, show that $\mathbb{E}(S_n - \mathbb{E}S_n)^3 = \sum_{i,j,k=1}^n \mathbb{E}[(X_i - m)(X_j - m)(X_k - m)]$.
- Using independence, i.e. that $\mathbb{E}[\prod_{i=1}^n f_i(W_i)] = \prod_{i=1}^n \mathbb{E}[f_i(W_i)]$ for any functions $f_1, \ldots, f_n$ and any independent random variables $W_1, \ldots, W_n$, show that $\mathbb{E}[(X_i - m)(X_j - m)(X_k - m)] = 0$ unless i, j, k are all the same. (Note that for any random variable Y, $\mathbb{E}(Y - \mathbb{E}(Y)) = 0$.)
- Deduce that $\mathbb{E}(S_n - \mathbb{E}S_n)^3 = n\mathbb{E}(X_1 - \mathbb{E}X_1)^3$.
- Now compute $\mathrm{skw}(S_n) = \frac{\mathbb{E}(S_n - \mathbb{E}S_n)^3}{\mathrm{Var}(S_n)^{3/2}}$ in terms of $\mathrm{skw}(X_1)$.

(3) Suppose $X \sim \mathrm{Bern}(p)$. Show that $\mathrm{skw}(X) = \frac{1-2p}{\sqrt{p(1-p)}}$ by direct computation.

(4) Suppose $X \sim \mathrm{Bin}(n,p)$. Show that $\mathrm{skw}(X) = \frac{1-2p}{\sqrt{np(1-p)}}$, so that it vanishes as $N \to \infty$. (In particular, this shows that averaging a bunch of random variables can reduce skewness.)

1.6. **Some more computations.** Keep the notation in the setting of Problem 1.5. Define the *kurtosis* of X by $\mathrm{kur}(X) = \frac{\mathbb{E}(X-m_1)^4}{\sigma^4}$. (This is kind of like a variance, but it tells you a little more about the shape of the graph of the pdf.)

(1) Show that if $X \sim N(\mu, \sigma^2)$, then $\mathrm{kur}(X) = 3$. Notice how this is much simpler! (It does not depend on the parameters of the distribution.)

(2) Let X_1, X_2 be i.i.d. $N(0,1)$. Define $S = X_1 + X_2$. *Without using the fact that* $X_1 + X_2 \sim N(0,2)$, show that $\mathrm{kur}(S) = 3$. (In particular, use $\mathrm{kur}(S) = \frac{\mathbb{E}(S - \mathbb{E}S)^4}{\mathrm{Var}(S)^2}$.)